THE UNIVERSITY OF CHICAGO

WEAK AND STRONG NORMALIZATION OF TIERED PURE TYPE SYSTEMS VIA TYPE-PRESERVING TRANSLATION

A DISSERTATION SUBMITTED TO
THE FACULTY OF THE DIVISION OF THE PHYSICAL SCIENCES
IN CANDIDACY FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

DEPARTMENT OF COMPUTER SCIENCE

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CHICAGO, ILLINOIS
MAY 16, 2023

To Mom and Dad

You may forget but

let me tell you
this: someone in
some future time
will think of us

-Sappho

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ACKNOWLEDGMENTS

I owe the greatest deal to my family for their support throughout my entire life; to my advisor Stuart Kurtz for our weekly chats that went on during the entire back half of my graduate career; to Alexander Razborov, who taught me much about research when I first came to the university. I extend this appreciation to my office mates for maintaining morale in those familiar times of melodramatic academic turmoil, to those professors at the university who made my time at the university what it was, and to those friends who made Chicago what is was. And for the obgliatory list of names (in a particular order): Anteres Chen, Galen Harrison, Mourad Heddaya, Neng Huang, Jafar Jafarov, Chris Jones, Marie Kim, Will Kong, Matt McPartlon, JQ Lee, Annie Lin, Matt Teichman, Andy Yang. Thank you all.

ABSTRACT

A type system is weakly normalizing if every typable expression has a normal form, and is strongly normalizing if no typable expression appears in an infinite reduction sequence. The Barendregt-Geuvers-Klop conjecture asks if weak normalization implies strong normalization for all *pure type systems*, a class of systems which generalizes the λ-cube. There are two natural techniques based on type-preserving translations for proving strong normalization from weak normalization. The first is to define a translation from expressions to I-expressions (*i.e.*, expressions in which λ-bound variables *must* be used). Strong normalization then readily follows from weak normalization by a conservation result along the lines of the one proved by Church [1941] for the untyped λ-calculus. This was first done by Xi [1996] and Sørensen [1997] for a subset of the λ-cube, and was generalized to a class of pure type systems by Barthe et al. [2001]. The second technique is to define an infinite-reduction-path-preserving translation to a weak sub-system for which the conjecture is known to hold. Then, by a straightforward boot-strapping argument, the conjecture can be shown to hold for the full system. Translations of this form were first presented by Geuvers and Nederhof [1991] and Harper et al. [1993] for systems in the λ-cube, and similar techniques were used by Roux and Doorn [2014] for a class of pure type systems.

These syntactic techniques have the further benefit that they are proof-theoretically weak, and so can be used to address the stronger form of the conjecture noted by Xi [1997], which asks if the proof of strong normalization from weak normalization can be carried out in Peano arithmetic, or even Heyting arithmetic. This is notable insofar as Girard [1972] proved that the normalization of the pure type system $\lambda 2$ implies the consistency of Peano arithemtic (within Peano arithmetic), so these results imply proof-theoretic results.

In what follows, I present three translations, one of the first form and two of the second form. The first is a generalization of the thunkification translation of Xi [1997], which maps expressions to I-expressions using a more fine-tuned padding scheme than the one used

by Barthe et al. [2001], leading to fewer technical restrictions on the class of systems to which it applies. The second is a generalization of the dependency-eliminating translation of Geuvers and Nederhof [1991] to a class of pure type systems which are known to be strongly normalizing, but for which the stronger form of the conjecture is demonstrated for the first time. The third is a novel application of ideas from Roux and Doorn [2014] (and, consequently, from Geuvers and Nederhof [1991]) for a class of pure type systems with "irrelevant" structure. These systems have properties like dependent types (albeit, a very weak notion of dependent types) which have not appeared in any of the classes of systems previously considered.

All of this work is done in the framework of *tiered pure type systems*, a simple class of persistent pure type systems which are concretely specified but sufficient to study in most cases regarding questions of normalization. I consider the introduction of this class of systems one of the more important contributions of this work; their combinatorial nature makes them easier to study than previously considered classes of systems. They encompass, in some sense, the simplest extension of the λ-cube, and have many desired meta-theoretic properties, but also leave much to be further developed.

CHAPTER 1

INTRODUCTION

1.1 Background

This dissertation is concerned with the relationship between weak and strong normalization of pure type systems, a class of typed λ-calculi. I'd like to begin by motivating said concern with a very brief (read: caricatured) historical outline of the relevant material. As much as I would like the following material to be based in first principles, the pedagogical problem of presenting type theory without first presenting a course-worth of material so far eludes me, and so I will be assuming general familiarity with basic logical concepts.

1.1.1 The Untyped Lambda Calculus

Church [1941] introduced what is now called the *untyped λ-calculus* as a theory of "functions as rules"[1] (*e.g.*, the function "given x, add 1 to x") as opposed to functions as graphs in the set-theoretic tradition (*e.g.*, the "function" $\{(x, 1+x) : x \in \mathbb{N}\}$). It is an equational theory over expressions from T, defined inductively as follows:

- $\mathsf{V} \subset \mathsf{T}$;
- $x \in \mathsf{V}$ and $M \in \mathsf{T}$ implies $\lambda x.\ M$;
- $M \in \mathsf{T}$ and $N \in \mathsf{T}$ implies $MN \in \mathsf{T}$;

where V is a fixed set of variable symbols.[2] The construct '$\lambda x.\ M$' is referred to as *abstraction* (over the variable x), and is thought of as yielding a function on x (*e.g.*, abstracting over x in the expression "add 1 to x" yields the function "given x, add 1 to x"). The construct 'MN'

1. This phrase is quoted by Barendregt [1984], who does not give a citation, but refers to it as "old fashioned," and so must have pulled it from the rarefied air of those times.

2. I take the usual liberty of being agnostic regarding the ontological status of "symbols."

is referred to as *application* (of one expression to another), and it has been noted by many (Barendregt [1984] included) that this is a mild misnomer, as application in this setting is a syntactic notion and should be thought of as yielding the application itself, not the value of applying the function to its argument (*e.g.*, "'given x, add 1 to x' applied to 2" as opposed to "3"). In subsequent exposition, I use Backus-Naur form grammars to describe inductively defined expressions, *e.g.*

$$\mathsf{T} ::= \mathsf{V} \mid \lambda\mathsf{V}.\ \mathsf{T} \mid \mathsf{T}\mathsf{T}$$

The theory has at its core an interest in *computation*—the notion of rules invoking more computational associations than graphs[3]—and describes a convertibility relation via a single computational rule which, for historical reasons, is called β:

$$(\lambda x.\ M)N = M[N/x]$$

where $M[N/x]$ is the result of substituting N for x in M (eliding the nasty details of substitution with which seasoned readers are keenly aware). This relation is lifted compatibly through the structure of expressions, *i.e.*, β-reductions can be done anywhere in the expression, not just at the top level; pictorially speaking:

$$(\ldots(\lambda x.\ M)N\ldots) = (\ldots M[N/x]\ldots)$$

Formally, the theory is the equational presentation of the compatible equivalence closure of β; viewed as a relation,

$$\beta = \{((\lambda x.\ M)N, M[N/x]) \mid M, N \in \mathsf{T}\}$$

3. It seems worth noting that students are so often confused when we refer to graphs as functions, since they look nothing like the functions of our imaginations, at least for the seeming majority of us.

and its compatibility closure '$\to_\beta$' is the smallest relation containing β such that $M \to_\beta N$ implies

$$\lambda x.\ M \to_\beta \lambda x.\ N$$

$$MP \to_\beta NP$$

$$PM \to_\beta PN$$

I have avoided up to this point discussing any tricky technicalities in the definition of this theory, but I will note now that expressions are considered equal *up to α-conversion*, or the renaming of bound variables (so, for example, $\lambda x.\ \lambda y.\ x = \lambda z.\ \lambda w.\ z$) and I will observe the *Barendregt variable convention*, which states that when writing down an expression, bound variables are chosen to be distinct from free variables (a standard *proviso* for logical syntax).

There is a rich line of work regarding the λ-calculus as a logical theory (see, *e.g.*, the encyclopedic text by Barendregt [1984]), but it is sufficient for the purposes of this dissertation to consider the λ-calculus as an *abstract reduction system*. For the other systems considered in this introduction, I will ignore the associated equational theory altogether, and focus strictly on the associated reduction system as well.

1.1.2 Abstract Reduction Systems

An **abstract reduction system** is simply a set X together with a binary relation R on X, with the "attitude"[4] of viewing R as a notion of reduction, *i.e.*, xRy means *x reduces to y*.

Notation and Terminology. Fix a reduction system (X, R).

- $\to_R$ stands for R.
- $\twoheadrightarrow_R^+$ stands for the transitive closure of R.

4. I recently discovered the nLab [2023] page for the notion of a "concept with an attitude" which is used to describe the added psychological structure implicit in a mathematical object. I quite like this term.

- $\twoheadrightarrow_R$ stands for the reflexive transitive closure of R, and $x \twoheadrightarrow_R y$ is read "x reduces to y."

- $=_R$ stands for the equivalence closure of R.

- A **reduction sequence** is a potentially infinite sequence of elements $x_1, x_2, \ldots$ such that

$$x_1 \to_R x_2 \to_R \cdots$$

  It is straightforward to verify that $x \twoheadrightarrow_R y$ if there is a finite reduction sequence starting at x and ending at y, and that $x \twoheadrightarrow_R^+ y$ if the sequence is non-empty.

- An element y is an R-**normal form** (or is in R-normal form) if it is irreducible, *i.e.*, there is no element z such that yRz. It is an R-normal form of x if $x \twoheadrightarrow_R y$.

Properties. There are multitudes one may wish to verify of a given reduction system, but I restrict myself to the core four of interest here (and really, only the last two will play a substantial role).

- A system has the **Church-Rosser property** if $x =_R y$ implies there is an element z such that $x \twoheadrightarrow_R z$ and $y \twoheadrightarrow_R z$.

- A system has **unique normal forms** if $x \twoheadrightarrow_\beta y$ and $x \twoheadrightarrow_\beta z$ and both y and z are normal forms implies $y = z$.

- A system is **weak normalizing** (or has the weak normalization property) if every element of has an R-normal form. A particular element is weakly normalizing if it has an R-normal form.

- A system is **strong normalization** (or has the strong normalization property) if no element appears in an infinite reduction sequence. A particular element is strongly normalizing if it does not appear in an reduction sequence.

The study of abstract reduction systems postdates the study of the λ-calculus, with $(\mathsf{T}, \to_\beta)$ being an oft cited example of a prototypical reduction system (see, *e.g.*, the exposition of Bezem et al. [2003]). Church and Rosser [1936] proved that this system satisfies the first eponymous property, as well as the second (up to α-equivalence). From this, it is possible to show that normalizing β-equivalent expressions reduce to the same normal form, which indicates that β-reduction captures a reasonable notion of computation. The last two properties—which are the focus of this dissertation—do not hold of $(\mathsf{T}, \to_\beta)$. The expression

$$\Omega \triangleq (\lambda x.\ xx)(\lambda x.\ xx)$$

has no normal form since $\Omega \to_\beta \Omega$ and this is the only reduction that can be performed on Ω. Even the sub-reduction-system restricted to the weakly normalizing expressions is not strongly normalizing; given

$$I \triangleq \lambda x.\ x$$
$$K \triangleq \lambda x.\ \lambda y.\ x$$

$KI\Omega \to_\beta I$ and I is a normal form but $KI\Omega \to_\beta KI\Omega \to_\beta KI\Omega \ldots$

1.1.3 The Lambda-I Calculus

The previous remark is perhaps unsurprising; it seems a mere platitude to note that strong normalization is a stronger notion than weak normalization. But it is natural to consider under what conditions it is not.

The λI-calculus is the fragment of the λ-calculus in which every expression of the form $\lambda x.\ M$ is required to have x appear in M. I will use $\mathsf{T}_{\lambda I}$ to denote the set of such expressions. The expression K, for example, is not in $\mathsf{T}_{\lambda I}$, but Ω is (in particular, there are non-normalizing expressions in $\mathsf{T}_{\lambda I}$). The reason $KI\Omega$ is weakly normalizing but not strongly normalizing is that K can "throw away" its non-normalizing argument, which depends heavily on the fact that K is not in $\mathsf{T}_{\lambda I}$. Church [1941] proves that the inability to do this is a sufficient condition for weak and strong normalization to coincide.

Theorem 1. *(Church [1941], Conservation for λI) For any λI-expression M, if M is weakly normalizing then M is strongly normalizing.*

This is, in a loose sense, a reverse mathematical result. It may be equivalently stated that every λI-expression is either non-normalizing or strongly normalizing, but the more interesting reading (to me) is that weak normalizing is a sufficient assumption for proving strong normalization. So from the perspective of reduction systems, and with an eye towards what is to come, in order to show that a sub-system of $(T_{\lambda I}, \rightarrow_\beta)$[5] is strongly normalizing, it is sufficient to show that it is weakly normalizing.

It may be of some interest to the historically inclined that λI-calculus was the first calculus presented by Church [1941], but the calculus as we known it now (sometimes referred to as the λK-calculus) became the dominant form. Barendregt [1984] (pp. 38) cites some potential reasons for the shift.

1.1.4 The Simply Typed Lambda Calculus

Church [1940] introduced the simply typed λ-calculus—here on out denoted by $\lambda_\rightarrow$—as an extension of the simple types of Russell and Whitehead [1910, 1912, 1913] to include notions from the λ-calculus. In its modern form (and as it is presented here) we have a new grammar of *types*

$$\mathsf{Ty} ::= \mathsf{B} \mid \mathsf{Ty} \rightarrow \mathsf{Ty}$$

where B is a set of base types. The construct '$A \rightarrow B$' is a *function type* and is used to describe expressions which *look* like functions. In the tradition of Church-style typing, we

5. Here and in future iterations, I allow the reduction relation to be defined on a super-set, so as not to require notation for restricting the relation.

work over a slightly different expressions grammar

$$\mathsf{T} ::= \mathsf{V} \mid \lambda \mathsf{V}^{\mathsf{Ty}}.\ \mathsf{T} \mid \mathsf{TT}$$

in which abstracted variables are annotated with types.[6] Judgments of the form

$$x_1 : A_1, \ldots, x_k : A_k \vdash M : A$$

read as

"x_1 is of type A_1, and ... and, x_k is of type A_k implies M is of type A"

are then used to delineate a class of "well-behaved" expressions. Roughly speaking, types acts as a simple syntactic description of intended semantic behavior of their expressions. The rules for building typing judgments are as one might expect if we understand function types to delineate the expressions we presume to look and behave like functions.

Typing Judgments. A variable x is **fresh** if it does not appear in on the left side of the turnstile.

- **Start.** For any type A (*i.e.*, $A \in \mathsf{Ty}$) and any variable x

$$x : A \vdash x : A$$

- **Variable Introduction.** For any type B and variable x which is fresh with respect to Γ

$$\frac{\Gamma \vdash M : A}{\Gamma, x : B \vdash x : B}$$

- **Weakening.** For any type B and variables x which is fresh with respect to Γ

6. There is also a notion of Curry-style typing. This is sometimes called "domain-free" and the former "domain-full." See the work of Barthe and Sørensen [2000] for more details.

$$\frac{\Gamma \vdash M : A}{\Gamma, x : B \vdash M : A}$$

- **Abstraction.**

$$\frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x^{A}.\ M : A \to B}$$

- **Application.**

$$\frac{\Gamma \vdash M : A \to B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$

The rules as presented are a bit non-standard, but better mirror the shape of the rules for pure type systems considered in the next section. Despite this, I hope that they represent reasonable intuitions about how functions should behave. The application rule, for example, reads "if M is a function of type $A \to B$, and N is of type A, then it is well-defined to apply M to N and we should expect the result to be of type B." For a more concrete example, if $\mathsf{Nat} \in \mathsf{B}$, then it is possible to derive

$$Z : \mathsf{Nat}, S : \mathsf{Nat} \to \mathsf{Nat}, A : \mathsf{Nat} \to (\mathsf{Nat} \to \mathsf{Nat}) \vdash \lambda x^{\mathsf{Nat}}.\ A(SZ)x : \mathsf{Nat} \to \mathsf{Nat}$$

a slightly obfuscated but potentially familiar example.

Given these typing rules, we can then consider the rewrite system $(\mathsf{T}_{\lambda_\to}, \to_\beta)$ where $\mathsf{T}_{\lambda_\to}$ is the subset of expressions which are typable according to the above rules (formally, M is **typable** if there is a type A and a context Γ such that $\Gamma \vdash M : A$ is derivable). Gandy [1980] notes that his advisor Alan Turing—in a letter he wrote in the 1940s—was the first to prove that $\lambda_\to$ is weakly normalizing via a measure on expressions which is monotonically decreasing in a particular reduction strategy. It would take several decades before Tait [1967] proved it is strongly normalizing via a more complicated proof which, in essence, uses logical relations as described by Plotkin [1973]. Xi [1996] and Sørensen [1997] recognized that, since the proof of weak normalization is a fair bit simpler, it is useful to derive strong normalization from weak normalization, and both achieve this via a translation into the λI-calculus. Again, we have a reverse mathematical question: is weak normalization of $(\mathsf{T}_{\lambda_{\to}}, \to_\beta)$ a sufficient condition to more easily prove its strong normalization?

1.1.5 Beyond Simple Types

In the unlikely event that I have piqued the interest of a non-expert, I recommend at this point reading any of the great introductory texts in the field (*Type Theory and Formal Proof* by Nederpelt and Geuvers [2014] is a favorite of mine). What remains will be informal and cursory.

The decades after the introduction of simple types saw an explosion of type theories for logic, linguistics, computer science, and many other sister subjects. Type systems may be fancified with a number of new constructions, shapes, annotations, *etc.*, but there were three features of particular importance during this development, which were noted by Barendregt [1991] to share the property that they blurred the line between types and terms.

- **Polymorphism.** *(or abstraction over type variables in terms)* This allows for functions which are, in a sense, agnostic to the type of their argument. *Example.* given Type, the type of types, the expression $\lambda A^{\mathsf{Type}}.\ \lambda x^A.\ x$ can be applied to different types to yield differently typed identity functions.[7]

- **Type Constructors.** *(or abstraction over type variables in types)* This allows for functions that can return types themselves. *Example.* given an empty type $\bot$, it may be useful to write the function $\lambda A^{\mathsf{Type}}.\ A \to \bot$, which returns the negated form the type, in the sense of the *Curry-Howard isomorphism*.[8]

- **Dependent Types.** *(or abstraction over term variables in types)* This allows for the

7. A typical mouth-full when discussing impredicativity.

8. The Curry-Howard isomorphism says that types can be read as mathematical statements and terms can be read as proofs. See for details.

definition of predicates whose types are of the form $A \to \mathsf{Type}$ and whose inhabiting terms are evidence that the predicate holds on its argument (again, in the sense of the Curry-Howard isomorphism). *Example.* given a definition of natural numbers, it is possible to define $\mathsf{even} : \mathsf{Nat} \to \mathsf{Type}$ where $\mathsf{even}\ 2$ represents "2 is even."

These three features form the "basis" of a class of eight systems called the λ-cube, introduced by Barendregt [1991], which uniformly describes these three features in single framework. The systems in the λ-cube vary in which of the three above features they have (hence, the cube structure).

System	Polymorphism?	Type Constructors?	Dependent Types?
$\lambda_{\to}$	no	no	no
$\lambda 2$	yes	no	no
$\lambda \underline{\omega}$	no	yes	no
$\lambda \omega$	yes	yes	no
λP	no	no	yes
$\lambda P2$	yes	no	yes
$\lambda P\underline{\omega}$	no	yes	yes
λC	yes	yes	yes

The most important systems in the λ-cube, by name, are

- ($\lambda_{\to}$) simply typed lambda calculus, introduced by Church [1940].
- ($\lambda 2$) second-order lambda caluclus, introduced independently by Girard [1972] and Reynolds [1974].
- ($\lambda \omega$) higher-order polymorphic lambda calculus, introduced by Girard [1972].
- (λP) logical framework (roughly speaking), based on ideas of De Bruijn [1968] and perhaps most notably implemented by Harper et al. [1993].

- (λC) calculus of constructions, introduced by Coquand and Huet [1986].

I choose not to present them formally here, in part because the are so similar to the *pure type systems* of the next section. I instead refer the reader to the survey by Barendregt [1993].

Every system in the λ-cube is strongly normalizing. As mentioned above, Tait [1967] proved strong normalization of $\lambda_{\rightarrow}$. Girard [1972] proved strong normalization of $\lambda 2$ and $\lambda\omega$ by an extension of Tait's method. The exact first proof of strong normalization of λC is tricky to pin down, because a number of the first proofs had mistakes; Coquand and Gallier [1990] (Section 7) give a fair timeline of these results. The proof of strong normalization of λC which is most relevant here is the one by Geuvers and Nederhof [1991], which shows strong normalization of λC be defining an infinite-reduction-path-preserving translation from λC ot $\lambda\omega$ and then bootstrapping with strong normalization with $\lambda\omega$. Harper et al. [1993] use a similar result to prove strong normalization of λP, which is translated down to $\lambda_{\rightarrow}$.

Beyond the λ-cube the picture is much less clear. The systems λU and λU^{-}, which are mild extension of $\lambda\omega$, were shown by Girard [1972] and Coquand [1995] to be non-weakly normalizing. On the other hand, Luo [1990] presented an extention of λC with an infinite heirarchy of universes call the *extended calculus of constructions* (denote by ECC) and showed it is strongly normalizing. The class of pure type systems captures these extensions along with a slew of others, about many of which normalization is unknown.

At this point, I think it appropriate to transition to more formal definitions. Note that I give a statement of the problem considered in this dissertation, as well as the contributions made, in Section 1.3.

1.2 Definitions

The class of pure type systems is the basis of a very general framework for describing type systems and their meta-theory. These systems vary in their sort structure and their product type formation rules, and include the entire λ-cube. Barendregt cites Berardi [1988] and

Terlouw [1989] for their conception, though Geuvers and Nederhof [1991] are cited as having given the first explicit definition, based on the previous two works. The presentations of Barendregt [1991, 1993] are perhaps the best known sources.[9]

A pure type system is specified by a triple of sets $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ satisfying $\mathcal{A} \subset \mathcal{S} \times \mathcal{S}$ and $\mathcal{R} \subset \mathcal{S} \times \mathcal{S} \times \mathcal{S}$. The elements of $\mathcal{S}$, $\mathcal{A}$, and $\mathcal{R}$ are called sorts, axioms and rules, respectively. I use s and t as meta-variables for sorts.[10]

For each sort s, fix a $\mathbb{Z}^+$-indexed set of expression variables V_s. Let ${}^s v_i$ denote the ith expression variable in V_s and let V denote $\bigcup_{s \in \mathcal{S}} \mathsf{V}_s$. I use x, y, and z as meta-variables for expression variables. The choice to annotate variables with sorts is one of convenience. The annotations can be dropped for the systems I consider, and are selectively included in the exposition. This observation regarding variable annotations was first made by Geuvers [1993] (Definition 4.2.9).

1.2.1 Expressions

The set of expressions of a pure type system with sorts $\mathcal{S}$ is described by the grammar

$$\mathsf{T} ::= \mathcal{S} \mid \mathsf{V} \mid \Pi \mathsf{V}^{\mathsf{T}}.\ \mathsf{T} \mid \lambda \mathsf{V}^{\mathsf{T}}.\ \mathsf{T} \mid \mathsf{T}\mathsf{T}$$

I use capital Modern English letters like M, N, P, Q, A, B, and C as meta-variables for expressions.

The **sub-expressions** of a given expression are defined by the function $\mathsf{sub} : \mathsf{T} \to 2^{\mathsf{T}}$

9. I am of the opinion that, after the development of the lambda cube, the notion of pure type systems was soon to follow, and that all aforementioned should be cited as originators.

10. For any subsequent meta-variables, I use positive integer subscripts and tick marks, *e.g.*, s_1, s_2, and s'. Note, however, that in later sections, s_i will refer to a particular sort in tiered systems. I will try to be as clear as possible when distinguishing between these two cases of notation.

inductively.

$$\begin{aligned}
\mathsf{sub}(s_j) &\triangleq \{s_j\} \\
\mathsf{sub}({}^{s_j}x) &\triangleq \{{}^{s_j}x\} \\
\mathsf{sub}(\Pi x^A.\ B) &\triangleq \mathsf{sub}(A) \cup \mathsf{sub}(B) \cup \{\Pi x^A.\ B\} \\
\mathsf{sub}(\lambda x^A.\ M) &\triangleq \mathsf{sub}(A) \cup \mathsf{sub}(M) \cup \{\lambda x^A.\ M\} \\
\mathsf{sub}(MN) &\triangleq \mathsf{sub}(M) \cup \mathsf{sub}(N)
\end{aligned}$$

I will write $N \subset M$ for $N \in \mathsf{sub}(M)$.

'Π' and 'λ' bind variables in their respective terms. The **free variables** of an expression are defined by the function $\mathsf{FV} : \mathsf{T} \to 2^{\mathsf{V}}$ inductively.

$$\begin{aligned}
\mathsf{FV}(s_i) &\triangleq \emptyset \\
\mathsf{FV}(x) &\triangleq \{x\} \\
\mathsf{FV}(\Pi x^A.\ B) &\triangleq \mathsf{FV}(A) \cup \mathsf{FV}(B) \setminus \{x\} \\
\mathsf{FV}(\lambda x^A.\ B) &\triangleq \mathsf{FV}(A) \cup \mathsf{FV}(B) \setminus \{x\} \\
\mathsf{FV}(MN) &\triangleq \mathsf{FV}(M) \cup \mathsf{FV}(N)
\end{aligned}$$

The **compatible closure** of a relation R on expressions, typically written $\to_R$ is the smallest relation containing R such that $M \to_R N$ implies

$$\begin{array}{cc}
\Pi x^M.\ P \to_R \Pi x^N.\ P & \Pi x^P.\ M \to_R \Pi x^P.\ N \\
\lambda x^M.\ P \to_R \lambda x^N.\ P & \lambda x^P.\ M \to_R \lambda x^P.\ N \\
MP \to_R NP & PM \to_R PN
\end{array}$$

The **renaming of the free variable** x **to** y is defined as follows.

$$
\begin{aligned}
s_j[y/x]^r &\triangleq s_j \\
z[y/x]^r &\triangleq \begin{cases} y & z = x \\ z & \text{otherwise} \end{cases} \\
\Pi z^A.\ B[y/x]^r &\triangleq \begin{cases} \Pi z^{A[y/x]^r}.\ B & z = x \\ \Pi z^{A[y/x]^r}.\ B[y/x]^r & \text{otherwise} \end{cases} \\
\lambda z^A.\ M[y/x]^r &\triangleq \begin{cases} \lambda z^{A[y/x]^r}.\ M & z = x \\ \lambda z^{A[y/x]^r}.\ M[y/x]^r & \text{otherwise} \end{cases} \\
MN[y/x]^r &\triangleq M[y/x]^r N[y/x]^r
\end{aligned}
$$

Define the relation α as:

$$
\begin{aligned}
\Pi x^A.\ B &\quad \alpha \quad \Pi y^A.\ B[y/x]^r \\
\lambda x^A.\ M &\quad \alpha \quad \lambda y^A.\ M[y/x]^r
\end{aligned}
$$

for all expressions A, B, and M and variables x and y, where $y \notin \mathsf{FV}(\Pi x^A.\ B)$ in the first case and $y \notin \mathsf{FV}(\lambda x^A.\ M)$ in the second case. Define α**-equivalence**, denote by '$=_\alpha$', to be the compatible equivalence closure of α. All expressions are considered up to α-equivalence.

The **substitution of x with N** is defined as follows.

$$
\begin{aligned}
s_j[N/x] &\triangleq s_j \\
y[N/x] &\triangleq \begin{cases} N & y = x \\ y & \text{otherwise} \end{cases} \\
\Pi y^A.\ B[N/x] &\triangleq \Pi z^{A[N/x]}.\ B[z/y]^r[N/x] \qquad (\text{where } z \notin \mathsf{FV}(N) \cup \mathsf{FV}(B)) \\
\lambda y^A.\ M[N/x] &\triangleq \lambda z^{A[N/x]}.\ M[z/y]^r[N/x] \qquad (\text{where } z \notin \mathsf{FV}(N) \cup \mathsf{FV}(M)) \\
PQ[N/x] &\triangleq P[N/x]Q[N/x]
\end{aligned}
$$

Define the relation β as:

$$(\lambda x^A.\ M)N \quad \beta \quad M[N/x]$$

for any expressions A, M, and N, and any variable x. I will use the notation:

$$
\begin{array}{rl}
\to_\beta & \text{compatible closure of } \beta \\
\twoheadrightarrow_\beta^+ & \text{transitive closure of } \to_\beta \\
\twoheadrightarrow_\beta & \text{transitive reflexive closure of } \to_\beta \\
=_\beta & \text{equivalence closure of } \to_\beta
\end{array}
$$

An expression M **β-reduces** to an expression N if $M \twoheadrightarrow_\beta N$. I will not consider any other notions of reduction (*e.g.*, η) so I will often simply write of "reductions." A **reduction sequence** is a (possibly infinite) sequences of expression $M_1, M_2, \ldots$ such that

$$M_1 \to_\beta M_2 \to_\beta \ldots$$

An expression M is in **normal form** (or is a normal form) if there is no expression N such that $M \to_\beta N$. Alternatively: a **redex** of M is a sub-expression of the form $(\lambda x^A.\ P)Q$, and an expression is in normal form if it has no redexes. In order to more easily discuss redexes, we use **single-holed expressions**,[11] denoted by $C\langle\cdot\rangle$, which are defined inductively as follows.

- $\langle\cdot\rangle$ is a single-holed expression.
- For any variable x, expression M, and single-holed expression $C\langle\cdot\rangle$, the following are all single-holed expressions.
  - $\Pi x^{C\langle\cdot\rangle}.\ M$
  - $\Pi x^{M}.\ C\langle\cdot\rangle$
  - $\lambda x^{C\langle\cdot\rangle}.\ M$
  - $\lambda x^{M}.\ C\langle\cdot\rangle$
  - $C\langle\cdot\rangle M$
  - $M(C\langle\cdot\rangle)$

Let $C\langle M\rangle$ denote the expression resulting from replacing $\langle\cdot\rangle$ in $C\langle\cdot\rangle$ with M. The difference with variable substitution is that we do not concern ourselves with variable capture or renaming.

An expression M is **weakly normalizing** if there is some normal form N such that $M \twoheadrightarrow_\beta N$. It is **strongly normalizing** if it does not appear in any infinite reduction sequence. I will use the notation:

$$\mathsf{NF} \triangleq \{M \mid M \text{ is in normal form}\}$$

$$\mathsf{WN} \triangleq \{M \mid M \text{ is weakly normalizing}\}$$

$$\mathsf{SN} \triangleq \{M \mid M \text{ is strongly normalizing}\}$$

11. These are called "contexts" in other settings, but I've used a different name so as not to clash with the notion of contexts introduced below.

Just as in the case of the domain-free case, expressions in T satisfy the *Church-Rosser proprty* as well as the *unique normal forms property.* So by an abuse of notation, I will also use the function $\mathsf{NF} : \mathsf{WN} \to \mathsf{NF}$ which maps weakly normalizing expressions to their unique normal forms.

1.2.2 Typing Judgments

A **statement** is a pair of expressions, denoted by $M : A$. The first expression is called the **subject** and the second is called the **predicate**. A **proto-context** is a sequence of statements whose subjects are expression variables. The statements appearing in proto-contexts are called **declarations**. I use capital Greek letters like Γ, Δ, Φ, and Υ as meta-variables for contexts. Often the sequence braces of contexts are dropped and concatenation of contexts is denoted by comma-separation. The β-equality relation and substitution extend to contexts element-wise. For a context Γ and declaration $(x : A)$ I write $(x : A) \in \Gamma$ if that declaration appears in Γ, and $\Gamma \subset \Delta$ if $(x : A) \in \Gamma$ implies $(x : A) \in \Delta$. A **proto-judgment** is a proto-context together with statement, denoted $\Gamma \vdash M : N$. The designation "judgment" is reserved for proto-judgments that are derivable according to the rules below. Likewise, the designation "context" is reserved for proto-contexts that appear in some (derivable) judgment.

The pure type system $\lambda\mathcal{S}$ specified by $(\mathcal{S}, \mathcal{A}, \mathcal{R})$ has the following rules for deriving judgments. In what follows, the meta-variables s and s' range over all sorts in $\mathcal{S}$ when unspecified. A variable ${}^{s}x$ is **fresh** with respect to a context Γ if it does not appear anywhere in Γ.

- **Axioms.** For any axiom (s, s')

$$\vdash_{\lambda\mathcal{S}} s : s'$$

- **Variable Introduction.** For a variable ${}^{s}x$ which is fresh with respect to Γ

$$\frac{\Gamma \vdash_{\lambda\mathcal{S}} A : s}{\Gamma, {}^s x : A \vdash_{\lambda\mathcal{S}} {}^s x : A}$$

- **Weakening.** For a variable ${}^s x$ which is fresh with respect to Γ

$$\frac{\Gamma \vdash_{\lambda\mathcal{S}} M : A \qquad \Gamma \vdash_{\lambda\mathcal{S}} B : s}{\Gamma, {}^s x : B \vdash_{\lambda\mathcal{S}} M : A}$$

- **Product Type Formation/Generalization.** For any rule (s, s', s'')

$$\frac{\Gamma \vdash_{\lambda\mathcal{S}} A : s \qquad \Gamma, {}^s x : A \vdash_{\lambda\mathcal{S}} B : s'}{\Gamma \vdash_{\lambda\mathcal{S}} \Pi^s x^A.\ B : s''}$$

- **Abstraction.**

$$\frac{\Gamma, {}^s x : A \vdash_{\lambda\mathcal{S}} M : B \qquad \Gamma \vdash_{\lambda\mathcal{S}} \Pi^s x^A.\ B : s'}{\Gamma \vdash_{\lambda\mathcal{S}} \lambda^s x^A.\ M : \Pi^s x^A.\ B}$$

- **Application.**

$$\frac{\Gamma \vdash_{\lambda\mathcal{S}} M : \Pi^s x^A.\ B \qquad \Gamma \vdash_{\lambda\mathcal{S}} N : A}{\Gamma \vdash_{\lambda\mathcal{S}} MN : B[N/{}^s x]}$$

- **Conversion.** For any terms A and B such that $A =_\beta B$

$$\frac{\Gamma \vdash_{\lambda\mathcal{S}} M : A \qquad \Gamma \vdash_{\lambda\mathcal{S}} B : s}{\Gamma \vdash_{\lambda\mathcal{S}} M : B}$$

The subscript on the turnstile is dropped when there is no fear of ambiguity. The annotations on bound variables in Π-expressions and λ-expressions are non-standard, and will in most cases be dropped, but they are occasionally useful to maintain (*e.g.*, see Lemma 1). It is also standard to write $A \to B$ for $\Pi x^A.\ B$ in the case that x does not appear free in B, and to use the derived rule

$$\frac{\Gamma \vdash A : s \qquad \Gamma \vdash B : s'}{\Gamma \vdash A \to B : s''}$$

An expression M is said to be **derivable** in $\lambda\mathcal{S}$ if there is some context Γ and expression A such that $\Gamma \vdash_{\lambda\mathcal{S}} M : A$. Although there is no distinction between terms and types, it is useful to call a judgment a **type judgment** if it is of the form $\Gamma \vdash A : s$ where $s \in \mathcal{S}$, and a **term judgment** if it is of the form $\Gamma \vdash M : A$ where $\Gamma \vdash A : s$ for some sort s. I also write that M is a term and A is a type in this case. By type correctness (Lemma 3), a judgment that is not a type judgment is a term judgment, though some judgments are both type and term judgments. In the system specified by $(\{s_1, s_2\}, \{(s_1, s_2)\}, \emptyset)$, for example,

- $\vdash s_1 : s_2$ is a type judgements but not a term judgment,
- $x : s_1 \vdash x : s_1$ is a type judgment and a term judgment, and
- $x : s_1, y : x \vdash y : x$ is a term judgment but not a type judgment.

1.2.3 Meta-Theory

I collect here many of the standard the meta-theoretic lemmas necessary for the subsequent results. Much of the meta-theory of pure type systems was worked out by Geuvers and Nederhof [1991], and can be found in several of the great available resources on pure type systems (see the work of Barendregt [1993], Barthe et al. [2001], Geuvers [1993], Kamareddine et al. [2004], among others) so proofs are omitted. For the remainder of the section, fix a pure type system $\lambda\mathcal{S}$.

Lemma 1. *(Generation) For any context Γ and expression A, the following hold.*

- *Sort. For any sort s, if $\Gamma \vdash s : A$, then there is a sort s' such that $A =_\beta s'$ and $(s, s') \in \mathcal{A}$.*
- *Variable. For any sort s and variable ${}^s x$, if $\Gamma \vdash {}^s x : A$, then there is an type B such that $\Gamma \vdash B : s$ and $({}^s x : B)$ appears in Γ and $A =_\beta B$.*

- Π-*expression.* *For any sort s and expressions B and C, if*

$$\Gamma \vdash \Pi^s x^B.\ C : A$$

then there are sorts s', and s'' such that

$$\Gamma \vdash B : s \qquad \textit{and} \qquad \Gamma, {}^s x : B \vdash C : s'$$

and $(s, s', s'') \in \mathcal{R}$ and $A =_\beta s''$.

- λ-*expression.* *For any sort s and expressions B and M, if*

$$\Gamma \vdash \lambda^s x^B.\ M : A$$

then there is a type C and sort s' such that such that

$$\Gamma \vdash \Pi^s x^B.\ C : s' \qquad \textit{and} \qquad \Gamma, {}^s x : B \vdash M : C$$

and $A =_\beta \Pi^s x^B.\ C$.

- *Application.* *For expressions M and N, if $\Gamma \vdash MN : A$, then there is a sort s and types B and C such that $\Gamma \vdash M : \Pi^s x^B.\ C$ and $\Gamma \vdash N : B$ and $A =_\beta C[N/{}^s x]$.*

Lemma 2. *(Substitution) For contexts Γ and Δ and expressions M, N, A and B, if*

$$\Gamma, x : A, \Delta \vdash M : B \qquad \textit{and} \qquad \Gamma \vdash N : A$$

then

$$\Gamma, \Delta[N/x] \vdash M[N/x] : B[N/x].$$

Lemma 3. *(Type Correctness) For any context* Γ *and expressions* M *and* A*, if* $\Gamma \vdash M : A$ *then* $A \in \mathcal{S}$ *or there is a sort* s *such that* $\Gamma \vdash A : s$*.*

Lemma 4. *(Thinning) For contexts* Γ *and* Δ *and expressions* M *and* A*, if* $\Gamma \subset \Delta$ *and* $\Gamma \vdash M : A$*, then* $\Delta \vdash M : A$*.*

Lemma 5. *(Permutation) For contexts* Γ *and* Δ*, variables* x *and* y*, and expressions* A*,* B*,* M*, and* C*, if* x *does not appear free in* B *and*

$$\Gamma, x : A, y : B, \Delta \vdash M : C$$

then

$$\Gamma, y : B, x : A, \Delta \vdash M : C$$

Definition 1. *A pure type system is* ***functional*** *if*

- *if* $(s,t) \in \mathcal{A}$ *and* $(s,t') \in \mathcal{A}$ *then* $t = t'$*;*
- *if* $(s,t,u) \in \mathcal{R}$ *and* $(s,t,u') \in \mathcal{R}$*, then* $u = u'$*;*

Lemma 6. *(Type Unicity) If* $\lambda\mathcal{S}$ *is functional then for any context* Γ *and expressions* M*,* A*, and* B*, if* $\Gamma \vdash M : A$ *and* $\Gamma \vdash M : B$*, then* $A =_\beta B$*.*

Definition 2. *A sort* s *is a* ***top-sort*** *if there is no sort* s' *such that* $(s,s') \in \mathcal{A}$*. It is a* ***bottom-sort*** *if there is no sort* s' *such that* $(s',s) \in \mathcal{A}$*.*

Lemma 7. *(Top-Sort Lemma) For any context* Γ*, variable* x*, expressions* A *and* B*, and top-sort* s *the following hold.*

1. $\Gamma \not\vdash s : A$
2. $\Gamma \not\vdash x : s$
3. $\Gamma \not\vdash AB : s$

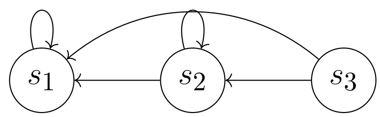


Figure 1.1: A visual representation of the system λU

1.2.4 Tiered Pure Type Systems

General pure type systems are notoriously difficult to work with so it is typical to consider classes of pure type systems satisfying certain properties, *e.g.*, persistence as subsequently defined.

Definition 3. *A pure type system $\lambda\mathcal{S}$ is* ***persistent*** *if it is functional (Definition 1) and*

- *if $(s,t) \in \mathcal{A}$ and $(s',t) \in \mathcal{A}$ then $s = s'$;*
- *$\mathcal{R}_{\lambda\mathcal{S}} \subset \{(s,s',s') \mid (s,s') \in \mathcal{S} \times \mathcal{S}\}$.*

From this point forward, I freely use the notation (s,s') for the rule (s,s',s'). One minor issue with properties like this is that it is often difficult to envisage the systems which satisfy them. In particular, the results tailored to a class of systems defined as such may use more meta-theoretic machinery than necessary. I choose, instead, to work with a simple class of systems I call *tiered* pure type systems, which have a very concrete description.

Definition 4. *Let n be a non-negative integer. A pure type system is* ***n-tiered*** *if its has the form*

$$\mathcal{S} = \{s_i \mid i \in [n]\}$$
$$\mathcal{A} = \{(s_i, s_{i+1}) \mid i \in [n-1]\}$$
$$\mathcal{R} \subset \{(s,s',s') \mid (s,s') \in \mathcal{S} \times \mathcal{S}\}$$

where $[n] \triangleq \{1,\ldots,n\}$.

A couple remarks about these systems:

- these systems can be envisaged as graphs as in Figure 1.1, which is a visual representation of the 3-tiered system λU. In such representations, an arrow (s_i, s_j) indicates the presence of the rule (s_i, s_j). Axioms are not represented in the graph except in the ordered the nodes are presented;

- the only 0-tiered system is the empty pure type system; there are two 1-tiered systems, specified by either $(\{s_1\}, \emptyset, \emptyset)$ or $(\{s_1\}, \emptyset, \{(s_1, s_1)\})$, neither of which have derivable expressions; the 2-tiered systems which contain the rule (s_1, s_1) are exactly the lambda cube;

- the n-tiered systems are considered in passing by Barthe et al. [2001] (Remark 2.39). They include natural subsystems of ECC^n (as defined by Luo [1990]) with only the two-sorted rules;

A large class of natural pure type systems can be classified as disjoint unions of tiered systems. In order to state this equivalence, I work in the structural theory of pure type systems presented by Roux and Doorn [2014].

Definition 5. *For pure type systems $\lambda\mathcal{S}$ and $\lambda\mathcal{S}'$, the* ***disjoint union*** *$\lambda\mathcal{S} \sqcup \lambda\mathcal{S}'$ is specified by*

$$\mathcal{S}_{\lambda\mathcal{S}\sqcup\lambda\mathcal{S}'} \triangleq \mathcal{S}_{\lambda\mathcal{S}} \sqcup \mathcal{S}_{\lambda\mathcal{S}'}$$
$$\mathcal{A}_{\lambda\mathcal{S}\sqcup\lambda\mathcal{S}'} \triangleq \mathcal{A}_{\lambda\mathcal{S}} \cup \mathcal{A}_{\lambda\mathcal{S}'}$$
$$\mathcal{R}_{\lambda\mathcal{S}\sqcup\lambda\mathcal{S}'} \triangleq \mathcal{R}_{\lambda\mathcal{S}} \cup \mathcal{R}_{\lambda\mathcal{S}'}$$

We begin by noting some basic facts about persistent systems. First, let $\mathcal{A}(s)$ denote the set of axioms in which s appears, *i.e.*, $\{(s', t') \in \mathcal{A} \mid s' = s \text{ or } t' = s\}$. Alternatively, viewing $\mathcal{A}$ as a graph, this is the set of edges in the neighborhood graph of s. If $\lambda\mathcal{S}$ is persistent, then $|\mathcal{A}(s)| \leq 2$ for any sort s. Let '$<_{\mathcal{A}}$', '$\leq_{\mathcal{A}}$', and '$\approx_{\mathcal{A}}$' denote the transitive, reflexive-transitive, and equivalence closure of $\mathcal{A}$, respectively. Equivalently, $s \leq_{\mathcal{A}} t$ if there is a finite sequence of sorts $t_1, \ldots, t_k$ such that $t_1 = s$ and $t_k = t$ and $(t_i, t_{i+1}) \in \mathcal{A}$ where $i \in [k-1]$. Call this a **sequential chain of length** $k-1$ from s to t (note that a sequential chain is not required to have distinct elements). If $\lambda\mathcal{S}$ is persistent, then this is the unique chain of length $k-1$ starting at s, or ending at t.

Proposition 1. *Let $\lambda\mathcal{S}$ be a persistent pure type system. For any sort s in $\mathcal{S}_{\lambda\mathcal{S}}$ and any k, there is at most one chain of length k starting at s, and likewise, at most one chain of length k ending at s.*

A class of systems which can be characterized by disjoint unions must be partitionable into atoms which can be analyzed individually.

Definition 6. *A pure type system $\lambda\mathcal{S}$ is **separable** if $(s, s') \in \mathcal{R}_{\lambda\mathcal{S}}$ implies $s \approx_{\mathcal{A}} s'$. It is **atomic** if $s \approx_{\mathcal{A}} s'$ for all sorts s and s'.*

There are, of course, many examples of important *non-separable* persistent pure type systems, *e.g.*, systems from the logic cube Barendregt [1991, 1993], Berardi [1990], Geuvers [1993, 1995] like Berardi's formulation of $\lambda\mathsf{PRED}\omega$ which is specified by

$$\mathcal{S} \triangleq \{*^s, \Box^s, *^p, \Box^p\}$$

$$\mathcal{A} \triangleq \{(*^s, \Box^s), (*^p, \Box^p)\}$$

$$\mathcal{R} \triangleq \{(*^p, *^p), (\Box^p, *^p), (\Box^p, \Box^p), (*^s, *^p), (*^s, \Box^p)\}$$

The rules $(*^s, *^p)$ and $(*^s, \Box^p)$ "cross" between two tiered systems. Despite this, there are also useful classes of systems which *are* separable, *e.g.*, generalized non-dependent systems are separable by *fiat*.

Definition 7. *Let $\lambda\mathcal{S}$ be a pure type system.*

- *$\lambda\mathcal{S}$ satisfies the **ascending chain condition** if $<_\mathcal{A}$ does, i.e., there is no infinite sequence of sorts $s, s', s'', \ldots$ such that $s < s' < s'' \ldots$; it satisfies the **descending chain condition** if there is no infinite sequence of sorts $s, s', s'', \ldots$ such that $s > s' > s'' \ldots$; it is **bounded** if it satisfies both the ascending and descending chain conditions.*

- *$\lambda\mathcal{S}$ is **weakly non-dependent** if $(s, s', s'') \in \mathcal{R}$ implies $s \geq s' \geq s''$.*

- *$\lambda\mathcal{S}$ is **stratified** if it satisfies the ascending chain condition and is weakly non-dependent.*

- *$\lambda\mathcal{S}$ is **generalized non-dependent** if it is stratified and persistent. If $\lambda\mathcal{S}$ is also bounded, I will write that it is **bounded non-dependent**. If $\lambda\mathcal{S}$ is also tiered, I will simply write that it is non-dependent.*

We can now characterize tiered systems in terms of the above properties.

Lemma 8. *A pure type system is tiered if and only if it is persistent, bounded, and atomic.*

Proof. It is straightforward to verify that tiered systems are persistent, bounded, and atomic, so I focus on the other direction. Suppose $\lambda\mathcal{S}$ is persistent, bounded, and atomic. If $\lambda\mathcal{S}$ is empty, then it is 0-tiered, so suppose $\lambda\mathcal{S}$ is non-empty. If $|\mathcal{S}| = 1$, then by boundedness, $\lambda\mathcal{S}$ is 1-tiered (in particular, $(s_1, s_1) \notin \mathcal{A}$), so suppose $|\mathcal{S}| \geq 2$. First, we prove that '$<_\mathcal{A}$' is a strict total order (and, hence, '$\leq_\mathcal{A}$' is a total order). If there is a positive length sequential chain from s to itself, then there is an infinite ascending chain, so '$<_\mathcal{A}$' is irreflexive. Transitivity is immediate. For connectedness, suppose s and t are sorts such that $s \neq t$. Note that '$\approx_\mathcal{A}$' can be viewed as the equivalence closure of '$<_\mathcal{A}$' and we can prove by induction on the definition that $s \approx t$ implies $s < t$ or $t < s$. The only interesting case is transitivity. Suppose there is sort s' such that $s \approx s'$ and $s' \approx t$. Then by the inductive hypothesis, $s < s'$ or $s' < s$ and $s' < t$ or $t < s'$. The cases in which transitivity of '$<_\mathcal{A}$' is not immediately applicable are (1) $s < s'$ and $t < s'$ or (2) $s > s'$ and $t > s'$. In the first case, there are sequential chains from s to s' and from t to s', and by Proposition 1, one must be a suffix of the other. This yields a chain from s to t or from t to s, depending on the lengths of the chains from s to s' and t to t' (these lengths must be distinct since $s = t$ otherwise). The second case is similar.

Next, note that, by boundedness there must be a unique top-sort and a unique bottom-sort in $\mathcal{S}$ (every finite non-empty total order has a unique maximum element and unique minimum element). By Proposition 1, there is a unique chain from s to t. Since $\lambda\mathcal{S}$ is atomic, this sequence includes all sorts. By the bound on the number of axioms in which a sort can appear, this chain includes all axioms. By persistence, the rules are as required by the definition of tiered systems. □

It is then straightforward to lift this to disjoint unions of tiered systems.

Lemma 9. *A pure type system is persistent, bounded, and separable if and only if is the disjoint union of tiered pure type systems.*

Proof. Let $\lambda\mathcal{S}$ be a pure type system that is persistent, bounded, and separable and consider the partition $\mathcal{P}$ of $\mathcal{S}$ into $\approx_{\mathcal{A}}$-equivalence classes. Let $\mathcal{S}_p$ be such an equivalence class and let $\lambda\mathcal{S}_p$ denote the pure system specified by

$$\mathcal{S}_{\lambda\mathcal{S}_p} \triangleq \mathcal{S}_p$$

$$\mathcal{A}_{\lambda\mathcal{S}_p} \triangleq \mathcal{A}_{\lambda\mathcal{S}} \cap (\mathcal{S}_p \times \mathcal{S}_p)$$

$$\mathcal{R}_{\lambda\mathcal{S}_p} \triangleq \mathcal{R}_{\lambda\mathcal{S}} \cap (\mathcal{S}_p \times \mathcal{S}_p \times \mathcal{S}_p)$$

The system $\lambda\mathcal{S}_p$ is persistent and bounded because $\lambda\mathcal{S}$ is, and it is atomic by definition, so by Lemma 8 it is tiered. We can then view $\lambda\mathcal{S}$ as the system $\bigsqcup_{\mathcal{S}_p \in \mathcal{P}} \lambda\mathcal{S}_p$.[12] Note that all axioms are accounted for by *fiat* and all rules are accounted for by separability. □

12. Formally, they are isomorphic pure type systems. The definition of a pure type system homomorphism is as one might expect, see the definition of Geuvers [1993] (Definition 4.2.5)—which is also used by Roux and Doorn [2014]—for more details.

Corollary 1. *A pure type system is bounded non-dependent if and only if it is the disjoint union of non-dependent tiered pure types systems.*

Roux and Doorn [2014] show that the (strong) normalization of a disjoint union of pure type systems is equivalent to the (strong) normalization of each of its individual summands. So on questions of normalization regarding persistent, bounded, separable systems it suffices to consider tiered systems.

Proposition 2. *If weak normalization implies strong normalization for all tiered pure type systems, then the same is true for all persistent, bounded, separable pure type systems. In particular, if weak normalization implies strong normalization for all non-dependent pure type systems, then the same is true for all bounded non-dependent pure type systems.*

This all sits in a more general theory. There are two natural extensions of tiered systems. First, to infinite tiered systems. A $\mathbb{Z}^+$**-tiered pure type system** is one of the form

$$\mathcal{S} = \{s_i \mid i \in \mathbb{Z}^+\}$$
$$\mathcal{A} = \{(s_i, s_{i+1}) \mid i \in \mathbb{Z}^+\}$$
$$\mathcal{R} \subset \{(s, s', s') \mid (s, s') \in \mathcal{S} \times \mathcal{S}\}$$

A $\mathbb{Z}^-$**-tiered pure type system** is one of the form

$$\mathcal{S} = \{s_i \mid i \in \mathbb{Z}^-\}$$
$$\mathcal{A} = \{(s_i, s_{i+1}) \mid i \leq -2\}$$
$$\mathcal{R} \subset \{(s, s', s') \mid (s, s') \in \mathcal{S} \times \mathcal{S}\}$$

And a **$\mathbb{Z}$-tiered pure type system** is one of the form

$$\mathcal{S} = \{s_i \mid i \in \mathbb{Z}\}$$

$$\mathcal{A} = \{(s_i, s_{i+1}) \mid i \in \mathbb{Z}\}$$

$$\mathcal{R} \subset \{(s, s', s') \mid (s, s') \in \mathcal{S} \times \mathcal{S}\}$$

It is straightforward to verify that a pure type system is atomic and generalized non-dependent if it is finite n-tiered or $\mathbb{Z}^-$-tiered. Furthermore, we can make the following simple observation.

Proposition 3. *Let $\mathcal{C}$ be a set of tiered pure type systems with the following property: if $\lambda\mathcal{S}$ is an infinite tiered pure type system in $\mathcal{C}$, then for every finite tiered sub-system $\lambda\mathcal{S}'$ of $\lambda\mathcal{S}$, there is a finite tiered subsystem $\lambda\mathcal{S}''$ such that $\lambda\mathcal{S}' \subset \lambda\mathcal{S}'' \subset \lambda\mathcal{S}$ and $\lambda\mathcal{S}'' \in \mathcal{C}$. It then follows that every system in $\mathcal{C}$ is (strongly) normalizing if and only if every finite system in $\mathcal{C}$ is (strongly) normalizing.*

This is simply because derivations are finite, and must be constructable in some finite fragment. So we can restrict our focus to n-tiered systems for well-structured classes of systems.

Corollary 2. *If weak normalization implies strong normalization for all non-dependent n-tiered pure type systems, then the same is true for all generalized non-dependent pure type systems.*

The second extension is to **cyclic n-tiered pure type systems**, which are n-tiered systems with the additional axiom (s_n, s_1). The directed graph induced by $\mathcal{A}$ of a persistent pure type system has the property that every vertex has in-degree and out-degree at most 1. Such a graph is made up of connected components of cycles and lines (which may be length 0, *i.e.*, points). Thus, the most general version of the above fact can be stated as follows.

Proposition 4. *If weak normalization implies strong normalization for all cyclic and non-cyclic finite tiered pure type systems, then the same is true for all persistent separable pure type systems.*

Moving foward, when I write 'tiered' I mean the finite non-cyclic case.

I close this section with some useful features of tiered systems. One of the primary benefits of working in persistent systems in general (and tiered systems in particular) is that derivable expressions can be classified by the *level* in the system at which they are derivable. This property is shown by defining a degree measure on expressions and classifying expressions according to their degree. This result is due to Berardi [1990] and Geuvers and Nederhof [1991], and the presentation here roughly follows the same course.

Definition 8. *The* ***degree*** *of an expression is given by the following function* $\deg : \mathsf{T} \to \mathbb{N}$.

$$\begin{aligned}
\deg(s_i) &\triangleq i + 1 \\
\deg({}^{s_i}x) &\triangleq i - 1 \\
\deg(\Pi x^A.\ B) &\triangleq \deg(B) \\
\deg(\lambda x^A.\ M) &\triangleq \deg(M) \\
\deg(MN) &\triangleq \deg(M)
\end{aligned}$$

Let T_j *denote* $\{M \in \mathsf{T} \mid \deg(M) = j\}$ *and let* $\mathsf{T}_{\geq j}$ *denote* $\{M \in \mathsf{T} \mid \deg(M) \geq j\}$.

Lemma 10. *(Classification) Let* $\lambda\mathcal{S}$ *be an* n*-tiered pure type system. For any expression* A*, the following hold.*

- $\deg(A) = n + 1$ *if and only if* $A = s_n$.
- $\deg(A) = n$ *if and only if* $\Gamma \vdash_{\lambda\mathcal{S}} A : s_n$ *for some context* Γ.
- *For* $i \in [n-1]$*, we have* $\deg(A) = i$ *if and only if* $\Gamma \vdash_{\lambda\mathcal{S}} A : B$ *and* $\Gamma \vdash_{\lambda\mathcal{S}} B : s_{i+1}$ *for some context* Γ *and expression* B.

In particular, for context Γ *and expressions* M *and* A*, if* $\Gamma \vdash_{\lambda\mathcal{S}} M : A$ *then* $\deg(A) = \deg(M) + 1$*.*

Finally, some useful facts about degree. See the presentation by Barendregt [1993] for proofs in the 2-tiered case.

Lemma 11.

- *If* $\deg(B) = j - 1$ *then* $\deg(A[B/^{s_j}x]) = \deg(A)$*.*
- *If* $A \twoheadrightarrow_\beta B$*, then* $\deg(A) = \deg(B)$*.*

1.3 Outline

First, the enticingly simple conjecture which is the topic of this dissertation.

Conjecture 1. *For all pure type systems* $\lambda\mathcal{S}$*, if* $\lambda\mathcal{S}$ *is weakly normalizing, then* $\lambda\mathcal{S}$ *is strongly normalizing.*

This can be regarded as the same sort of game as above: considering the reduction system associated with typable expressions in a given pure type system, does the assumption of weak normalization aid in proving strong normalization of this reduction system?

This conjecture was first made by Geuvers [1993] is his PhD thesis. Barendregt is noted by Barthe et al. [2001] to have presented the conjecture at the *Second International Conference on Typed Lambda Calculi and Applications (1995)*, and Klop is the co-author of a preprint which includes the conjecture (Ketema et al. [2004], Conjecture 1.1); it has come to be known as the *Barendregt-Geuvers-Klop conjecture*.

This conjecture is motivated, in part, by the fact that, despite weak normalization clearly being the weaker of the two properties, it is often sufficient for proving other important meta-theoretic properties; that is, strong normalization does not yield much beyond the ability to be agnostic to the choice of reduction strategy. If the type system encodes a logic, then weak normalization is sufficient to for consistency. It is also sufficient in the proof of Girard [1972] that normalization of $\lambda 2$ implies the consistency of arthmetic. Normalization is required in many proofs of the deciability of type checking (see, *e.g.*, the references given by Barthe [1999]), but again weak normalization is sufficient. It is also worth noting that many proofs of normalization in the literature are, in fact, strong normalization proofs. Except in the case of $\lambda_{\rightarrow}$, its unclear if and when weak normalization is easier to prove than strong normalization.[13] Observations to this effect were also made by Geuvers in his PhD thesis.

That this conjecture remains unsolved leads me to believe it is motivated by two distinct questions.

- First, the same reverse mathematical question as above: is weak normalization a sufficiently strong assumption to prove strong normalization? This question takes on a more interesting role in this setting, as it is known that not all pure type systems are weakly normalizing. Furthermore, since Girard [1972] proved normalization of $\lambda 2$ implies the consistency of Peano Arithmetic, the reverse mathematical question can be better calibrated: *can the proof of strong normalization from weak normalization can be carried out in Peano arithmetic, or even Heyting arithmetic?* The import, of course, is that we cannot, in Peano arithemtic, prove this by first proving normalization. This conjecture was first noted by Xi [1996].

- Falsifying this conjecture would in essence require finding a system with an expression that is weakly normalizing but not strongly normalizing. What would such an expression look like? The expression $KI\Omega$ in the untyped λ-calculus hides a non-normalizing expression which could not be typable in a normalizing type system. There are examples of more complicated λ-expressions using fixed-points which have the property that every sub-expression is weakly normalizing, but it is unknown if such fixed-points

13. This sentiment is not quite correct contemporarily, with the greater occurence of *normalization-by-evaluation* proofs. See the Habilitationsschrift of Abel [2013] for an overview.

are typable (see, *e.g.*, the work of Geuvers and Verkoelen [2015]).

Beyond weak and strong normalization, this dissertation is concerned with *type-preserving translations*, which are used for type-theoretic conservation theorems. In particular, there are two natural techiques for proving strong normalization from weak normalization via type preserving translations.

- Translate expression into the I-fragment of the system, *i.e.*, the subsystem in which all λ-bound variables must be used in the bodies of their λ-expressions, in analogy with the λI-calculus. In the I-fragment, strong normalization readily follows from weak normalization by a conservation theorem akin to the one proved by Church [1941]. This was done by Xi [1996] and Sørensen [1997] via a continuation-passing-style (CPS) translation for $\lambda_{\rightarrow}$, $\lambda 2$ and $\lambda\omega$, the *non-dependent* side of the λ-cube. Barthe et al. [2001] generalize this result to a class of non-dependent pure type systems. It is actually still an open problem to give a direct translation of this form for any *dependent* system, even on the dependent side of the λ-cube, *e.g.*, for λC.

- Define an infinite-reduction-path from a pure type system to one of its sub-systems for which the conjecture is known to hold. It then follows by a boot-strapping argument that the conjecture holds for the super-system: if $\lambda\mathcal{S}$ is weakly normalizing, then its sub-system $\lambda\mathcal{S}'$ is weakly normalizing, which is then strongly normalizing by assumption, and so $\lambda\mathcal{S}$ is strongly normalizing via the path-preserving translation. As noted above, this was done by Geuvers and Nederhof [1991] for λC to $\lambda\omega$ and by Harper et al. [1993] for λP to $\lambda_{\rightarrow}$. The original purpose of both of these translations was a proof of strong normalization, but they can be used as black-box results for the same boot-strapping argument above. A version of this argument was generalized to a class of pure type systems by Roux and Doorn [2014].

In the following chapters, I present three translations, one of the first form and two of the second form. Below is a rough sketch of the results in each chapter.

- **Chapter 3.** I prove strong normalization from weak normalization for a class of non-dependent systems, which directly strengthens the result by Barthe et al. [2001]. Their result holds for *clean, negatable, generalized non-dependent* pure type systems, and is proved via a CPS translation injected with uninterpreted padding variables which can be used to ensure variables get used in the bodies of λ-expressions. Via a technical analysis of the reduction behavior, I eliminate the condition of negatability. Roughly speaking, negatablility is required in order for the translation to even be definable. Rather than enforcing this restriction, the translation I present maps to expressions that are derivable *in a stronger system* which *is* negatable. This comes at the cost of no longer being able to easily preserve weak or strong normalization; this has to be proved separately and combinatorially. I also weaken the condition of cleanliness via two new meta-theoretic lemmas for pure type systems. One characterizes the degree of sub-expressions, and the other acts as a kind of type-theoretic Skolemization lemma, which is used to give a finer-grained padding scheme.

- **Chapter 4.** I present a path-preserving translation from a pure type system to its *non-dependent restriction*, the system with all dependent rules removed. This is a direct generalization of the translation from λC to $\lambda\omega$ by Geuvers and Nederhof [1991] and the one from λP to $\lambda_{\rightarrow}$ by Harper et al. [1993]. The restriction on the class of systems to which this applies is unfortunately quite strong, and thus only applies to a class of systems which are known to be strongly normalizing. But because the translation is proof-theoretically simple, the argument can be carried out in Peano arithmetic, giving a proof of the strong form of the conjecture for a new class of systems.

- **Chapter 5.** I present a novel path-preserving translation based on that of Roux and Doorn [2014] from a pure type system to one of its subsystems in which some *irrelevant* rules are removed (these rules are irrelevant with respect to normalization, but not respect to derivablity). The translation leverages sparse inhabitation of certain types to essentially pre-reduces λ-expressions on those types, yielding an expression that no longer needs those rules to be derived. This extends the conjecture to a larger class of systems with dependent rules, non-negatable sorts.

As I conclude this introduction, I note that remarks about future work are scattered throughout the dissertation, in part because there is a very obvious open question remaining. But one form of future work which is not addressed below is *formalization*. The results below are technical, but they are also fairly formal. This makes them good candidates for being mechanically checked, which would greatly improve my trust in them. I leave this as one of the more important remaining pieces of forthcoming work to be done.

CHAPTER 2

WEAK AND STRONG NORMALIZATION OF GENERALIZED NON-DEPENDENT PURE TYPE SYSTEMS VIA THUNKIFICATION

2.1 Introduction

As noted in the previous chapter, one approach to deriving strong normalization from weak normalization is to pass through some form of the λI-calculus, the fragment of the λ-calculus in which λ-bound variables *must* be used. In this setting, strong normalization readily follows from weak normalization by a conservation theorem along the lines of the one proved by Church [1941]. The argument is roughly as follows: suppose a system $\lambda\mathcal{S}$ is weakly normalizing. Define a translation from expressions to I-expressions which preserves (1) typability in $\lambda\mathcal{S}$ and (2) infinite reduction paths. Translated expressions are weakly normalizing by being typable in $\lambda\mathcal{S}$ and, hence, are strongly normalizing by the conservation theorem. And since the translation preserves infinite reduction paths, the untranslated expressions are themselves strongly normalizing.

This technique was originally used by Xi [1996] and Sørensen [1997] via a continuation-passing-style (CPS) translation, and Xi [1997] subsequently presented an alternative proof via thunkification that is arguably simpler; rather than passing around continuations, expressions are thunkified and thunks are *padded* via uninterpreted variables with sub-expressions necessary for translating to I-expressions. Both are essentially typed versions of Klop's ι-tranlsation, though the thunkification translation is more direct. See the survey in the work of Gørtz et al. [2003] for many more details on the relationships between these translations and others.

Barthe et al. [2001] generalized Xi's and Sørensen's result to prove that weak normalization implies strong normalization in *generalized non-dependent clean negatable* pure type systems.[1] In a research report, I presented the analogous generalization for Xi's thunkification translation to the same class of systems. The primary contribution of this chapter is a strengthened version of the BHS result, proved via a non-standard variant of this thunkfication translation. The following points outline the technical contributions, as they pertain to dealing with negatability and cleanliness, respectively.

- In the informal outline of Xi's and Sørensen's technique above, point (1) is doing quite a bit of work; translated expressions are weakly normalizing only because *they are still derivable in* $\lambda\mathcal{S}$ which is weakly normalizing by assumption. The endomorphic nature of the translation is what requires negatability. This property ensures the system is expressive enough for thunkified (or CPS-ified) expressions to be typable. But it suffices to show that translated expressions are typable in an extension of $\lambda\mathcal{S}$ (which is expressive enough to type translated expressions) and even a *non-normalizing* extension of $\lambda\mathcal{S}$, if it is possible to give a *combinatorial* proof that (1) the translation preserves weak normalization *at the level of terms* and (2) the target system has the required conservation result for I-expressions. I give a such a combinatorial proof of weak normalization preservation and conservation for a variant of the translation which can be applied to non-negatable systems (see Definition 12).

- Cleanliness, the second major technical restriction set by BHS, ensures the padding variables mentioned above are, in fact, typable. Roughly speaking, expressions are padded using variables injected in the context of the form ($\mathsf{pad} : A \to \bot \to \bot$), where $\bot$ is the type of the thunk expression $\bullet$ (a distinguised variable). Given an expression N of type A, a thunkified expression M of type $\bot \to B$ can be evaluated as $M(\mathsf{pad} N \bullet)$, so that the expression N is padded into M. The challenge is that the type A may have variables which appear in the context, and abstraction and generalization

1. In subsequent exposition, I will refer to this result as the BHS result, or simply BHS, for Barthe, Hatcliff and Sørensen.

(*i.e.*, the formation of λ-expressions and Π-expressions, respectively) both remove variables from the context, potentially making A no longer typable. Cleanliness disallows abstractions and generalizations which remove variables that appear in the types of padding variables. For a variant of the translation presented here, I use a different scheme for typing padding variables which allows for more fine-grained control of these dependencies. This technique requires two technical lemmas which may be of independent interest. The first is, in essense, a type-theoretic Skolemization lemma, which allows for contexts to be permuted modulo additional generalizations (Corollary 3). So we can generalize over the variables that appear in the types of padding variables, eliminating the dependence at the level of variables in the context. The second is a generalization of Corollary 2.46 from BHS (Lemma 13), which characterizes the degree of sub-expressions, and which is also useful in determining when variables can commute with padding variables.

In what follows I present some preliminary material, which one of the aboved mentioned meta-theoretic lemmas, as well as an important lemma in the area of maximal reductions. I then present the generalized thunkification translation in two parts: one part for the type-level translation and one part for the term-level translation. The term-level translation is analyzed in two parts, the negatable and non-negatable cases. I present the main result (Theorem 2) in isolation (Section 2.3.3) for completeness.

2.2 Preliminaries

Outside of the preliminary material presented in the section on definitions in the introductory chapter (Section 1.2), I include this short section with an important lemma about non-dependent pure type systems, as well as a lemma about perpetual reduction.

One of the benefits of working with non-dependent systems is that sub-expressions of a given expression cannot have lower degree than the expression itself. This makes it possible to reverse induct on degree, which is an important proof technique for these systems. Typically, a simplified version of this fact suffices, which says that the degree of variables appearing in an expression must be at least that of the expression itself.

Lemma 12. *Let $\lambda\mathcal{S}$ be a non-dependent n-tiered pure type system. For any expression A, if a variable ${}^{s_j}x$ appears free in A, then $j > \deg(A)$.*

I present a slightly stronger version which is a generalization of Corollary 2.46 from BHS, and for which the previous lemma is a corollary. It captures exactly the possible degrees of sub-expressions, using the following definition for keeping track of potential dependencies. It is worth noting, in future sections, the similarity with cleanliness (Definition 17).

Definition 9. *Let $\lambda\mathcal{S}$ be a non-dependent n-tiered pure type system. Define set $\mathcal{D}_{\lambda\mathcal{S}}(i)$ be the smallest set satisfying the following properties.*

- *$i \in \mathcal{D}_{\lambda\mathcal{S}}(i)$;*
- *if $(s_j, s_i) \in \mathcal{R}_{\lambda\mathcal{S}}$ then $\mathcal{D}_{\lambda\mathcal{S}}(j) \subset \mathcal{D}_{\lambda\mathcal{S}}(i)$;*
- *if $(s_j, s_{i+1}) \in \mathcal{R}_{\lambda\mathcal{S}}$, then $\mathcal{D}_{\lambda\mathcal{S}}(j) \subset \mathcal{D}_{\lambda\mathcal{S}}(i)$.*

These sets are well-defined because they can be constructed by reverse induction on degree.

Lemma 13. *Let $\lambda\mathcal{S}$ be a non-dependent n-tiered pure type system. There is an expression M derivable in $\lambda\mathcal{S}$ such that $\deg(M) = i$ with a sub-expression N such that $\deg(N) = j$ if and only if $j \in \mathcal{D}_{\lambda\mathcal{S}}(i)$.*

Proof. First, the left direction, by reverse induction on degrees. I prove the stronger fact that M is a derivable type, *i.e.*, there is a context Γ and sort s_i such that $\Gamma \vdash M : s_i$. Note that $\mathcal{D}_{\lambda\mathcal{S}}(n) = \{n\}$ so it suffices that $\varnothing \vdash s_{n-1} : s_n$. For arbitrary i, there are three cases to consider. First suppose $j = i$. If $i > 1$, then $\varnothing \vdash s_{i-1} : s_i$ and otherwise $x : s_1 \vdash x : s_1$. Next suppose $j \in \mathcal{D}_{\lambda\mathcal{S}}(k)$ where $(s_k, s_i) \in \mathcal{R}_{\lambda\mathcal{S}}$. By the inductive hypothesis, there is context Γ and expressions A and B such that $\Gamma \vdash A : s_k$, where $B \subset A$ and $\deg(B) = j$. If $i > 1$, then

$$\Gamma \vdash A \to s_{i-1} : s_i$$

and otherwise

$$\Gamma, x : s_1 \vdash A \to x : s_1.$$

Finally suppose $j \in \mathcal{D}_{\lambda\mathcal{S}}(k)$ where $(s_k, s_{i+1}) \in \mathcal{R}_{\lambda\mathcal{S}}$. Let Γ, A, and B be as in the previous case. Then $\Gamma \vdash A \to s_i : s_{i+1}$. If $i > 1$, then $\Gamma, y : A, x : A \vdash s_{i-1} : s_i$ so we have

$$\Gamma, y : A \vdash (\lambda x^A.\ s_{i-1})y : s_i$$

and otherwise, $\Gamma, z : s_1, y : A, x : A \vdash z : s_1$ so we have

$$\Gamma, z : s_1, y : A \vdash (\lambda x^A.\ z)y : s_1$$

This concludes the left direction.

The other direction also follows by reverse induction on i, and then by induction on the structure of derivations. It is straightforward to verify that the sub-expressions of any derivable expression of degree n is also of degree n. So let M be an derivable expression such that $\deg(M) = i$. It will follow by induction on the structure of derivations that if N is a sub-expression of M then $\deg(N) \in \mathcal{D}_{\lambda\mathcal{S}}(i)$. I focus on the cases on which the definition of $\mathcal{D}_{\lambda\mathcal{S}}(i)$ depends.

Product Type Formation. Suppose the last inference is of the form

$$\frac{\Gamma \vdash A : s_j \qquad \Gamma, x : A \vdash B : s_i}{\Gamma \vdash \Pi x^A.\ B : s_i}$$

where $(s_j, s_i) \in \mathcal{R}_{\lambda\mathcal{S}}$. Let C be a sub-expression of A. If $j > i$, then $C \in \mathcal{D}_{\lambda\mathcal{S}}(j)$ by the first inductive hypothesis, which is a subset of $\mathcal{D}_{\lambda\mathcal{S}}(i)$ by assumption. If $j = i$, then $C \in \mathcal{D}_{\lambda\mathcal{S}}(i)$ by the second inductive hypothesis. The case in which C is a sub-expression of B is similar.

Abstraction. Suppose the last inference is of the form

$$\frac{\Gamma, {}^{s_j}x : A \vdash M : B \qquad \Gamma \vdash \Pi x^A.\ B : s_{i+1}}{\Gamma \vdash \lambda x^A.\ M : \Pi x^A.\ B}$$

where $(s_j, s_{i+1}) \in \mathcal{R}_{\lambda\mathcal{S}}$. If $C \subset A$, then $C \in \mathcal{D}_{\lambda\mathcal{S}}(j)$ by the first inductive hypothesis. If $C \subset M$, then $C \in \mathcal{D}_{\lambda\mathcal{S}}(i)$ by the second inductive hypothesis. □

The second lemma presented here useful result due to Lévy [1978], regarding maximal reductions. It will be used in subsequent sections to analyze the normalization behavior of translated expressions.

Definition 10. *Let $\mu : \mathsf{T} \to \mathbb{N} \cup \{\infty\}$ be the function which maps an expression M to the length of the longest reduction sequence starting at M (where $\mu(M) = \infty$ if M has an infinite reduction sequence).*

Lemma 14. *Lévy [1978] For any expressions A, M, $N_1, \dots, N_k$ and variable*

$$\mu((\lambda x^A.\ M)N_1 \dots N_k) \leq 1 + \mu(A) + \mu(N_1) + \mu(M[N_1/x]N_2 \dots N_k)$$

2.3 The Translation

The thunkification translation of Xi [1997] was introduced as a simpler approach to deriving strong normalization from weak normalization in typed lambda calculi, as compared to its CPS counterpart (presented by himself and Sørensen [1997]). Roughly speaking, rather than passing continuations, the translation pervasively thunkifies expressions, using uninterpreted padding variables to store sub-expressions in the evaluation of thunkified expressions. This allows for expressions to be mapped to I-expressions, where strong normalization is more readily proved from weak normalization (Lemma 15). The translation is, in a sense, a direct typed implementation of Klop's ι-translation Klop et al. [1993].

The CPS translation was generalized by BHS to *generalized non-dependent, clean, negatable* pure type systems Barthe et al. [2001]. In a research report, I extended Xi's translation by analogy, which yielded a simple alternative proof of the BHS result. In this section, I present a refinement of this translation which eliminates the requirement of negatability and weakens the requirement of cleanliness. Before presenting the translation, I give an overview of the approach.

One of the fundamental features of non-dependent systems is that expressions cannot depend on other expressions of lower degree (Lemma 13). It is, thus, natural to prove properties of derivable expressions by reverse induction on degree. This motivates the following definition.

Definition 11. *A non-dependent n-tiered pure type system is i-**secure** if $deg(M) \geq i$ implies $M \in \mathsf{SN}$ for all derivable expressions M.*

For the remainder of the section, **fix an n-tiered pure type system $\lambda\mathcal{S}$ and suppose that $\lambda\mathcal{S}$ is weakly normalizing.** We prove that $\lambda\mathcal{S}$ is i-secure for every i, reverse inductively, and in doing so define two families of translations, one for types $\{\rho_i\}_{i=1}^n$ and one for terms $\{\tau_i\}_{i=1}^n$, which is possible exactly because of this fundamental property of non-dependent systems; the degrees of terms are lower than the degrees of their types, so there is no need for mutual dependency between the type translations and the term translations.

We then prove the following implications for every expression of degree $i - 1$ (that is, terms whose types are degree i).

$$\begin{aligned} M \in \mathsf{WN} &\Rightarrow \tau_i(M) \in \mathsf{WN} \\ &\Rightarrow \tau_i(M) \in \mathsf{SN} \\ &\Rightarrow M \in \mathsf{SN}. \end{aligned}$$

The analysis at each level differs in character according to whether s_i is negatable.

Definition 12. *A sort s_i is **negatable** if $(s_i, s_i) \in \mathcal{R}_{\lambda\mathcal{S}}$. An n-tiered pure type system is i-**negatable** if s_i is negatable in $\lambda\mathcal{S}$.*

In the case that s_i is already negatable, τ_i preserve typability in the same system and the first implication follows by *fiat*. The second implication follows the fact that τ_i translates expressions to *I-expressions*, an appropriate generalization of the λI-calculus for this setting, and for which there is an analogous conservation lemma.

Definition 13. *An expression M is an **I-expression at level** j if the following hold.*

- $\deg(M) \geq j$
- *If $\lambda x^A.\ N \subset M$ and $\deg(\lambda x^A.\ N) = j$, then x appears free in N.*

Lemma 15. *(Barthe et al. [2001], Lemma 5.16) Let λS be a non-dependent tiered pure type system that is i-secure. Then for every derivable I-expression M at level $i - 1$, if $M \in \mathsf{WN}$ then $M \in \mathsf{SN}$.*

In the case s_i is not negatable, ρ_i and τ_i are allowed to translate into a negatable extension, but the first implications will need to be proved combinatorially (Lemma 32) and the second implication needs to be shown to still hold (Lemma 26). In particular, the above lemma requires that $\lambda\mathcal{S}$ be i-secure, so i-security needs to be preserved in the negatable extension. In both cases, the last implication follows from a standard argument that τ_i preserves infinite reduction paths (Lemma 34).

For the following two translations, **fix a level i and suppose that $\lambda\mathcal{S}$ is a weakly normalizing non-dependent i-secure n-tiered pure type system.** The subscript 'i' will be included in definitions, but will typically be dropped in the following exposition.

2.3.1 Type-Level Translation

We need a distinguished type $\bot_i$ for each sort s_i to stand for the type of thunks. In the case of s_1, this requires an additional type variable in the context.

Definition 14. *If* $i > 1$, *let* $\bot_i$ *denote* s_{i-1} *and, otherwise, let* $\bot_1$ *be a distinguished variable. Likewise, if* $i > 1$ *let* Δ_i *denote the empty context, and otherwise, let* Δ_1 *denote the context* $(\bot_1 : s_1)$.

The translation ρ gives the types of thunkified terms. This means pervasively replacing each type A where $\deg(A) = i$ with its corresponding thunkified form, *e.g.*, $\bot_i \to \rho_i(A)$.

Definition 15. *Define the functions*

$$\rho_i : \mathsf{T}_{\geq i} \to \mathsf{T}_{\geq i} \qquad \textit{and} \qquad \rho'_i : \mathsf{T}_{\geq i} \to \mathsf{T}_{\geq i}$$

simultaneously as follows.

$$\begin{aligned}
\rho_i(s_j) &\triangleq s_j \qquad \textit{(where } j \geq i-1\textit{)} \\
\rho_i({}^{s_j}x) &\triangleq {}^{s_j}x \qquad \textit{(where } j \geq i+1\textit{)} \\
\rho_i(\Pi^{s_j}x^A.\ B) &\triangleq \Pi^{s_j}x^{\rho'_i(A)}.\ \rho_i(B) \\
\rho_i(\lambda^{s_j}x^A.\ M) &\triangleq \lambda^{s_j}x^{\rho_i(A)}.\ \rho_i(M) \\
\rho_i(MN) &\triangleq \rho_i(M)\rho_i(N) \\
\rho'_i(A) &\triangleq \begin{cases} \bot_i \to \rho_i(A) & \deg(A) = i \\ \rho_i(A) & \textit{otherwise.} \end{cases}
\end{aligned}$$

The function ρ'_i *is extended to contexts as follows.*

$$\begin{aligned}
\rho'_i(\varnothing) &\triangleq \varnothing \\
\rho'_i(\Gamma, {}^{s_j}x : A) &\triangleq \begin{cases} \rho'_i(\Gamma), {}^{s_j}x : \rho'_i(A) & j \geq i \\ \rho'_i(\Gamma) & \textit{otherwise.} \end{cases}
\end{aligned}$$

The type-level translations are required to commute with substitution and preserve β-

equivalence. These two features are necessary for translating application and conversion inferences in the next section. The following two lemmas are standard.

Lemma 16. *(ρ and ρ' commute with substitution) For variable ${}^{s_j}x$ and expressions M and N where* $\deg(N) = j - 1$ *the following hold.*

- $\rho(M[N/{}^{s_j}x]) = \rho(M)[\rho(N)/{}^{s_j}x]$
- $\rho'(M[N/{}^{s_j}x]) = \rho'(M)[\rho(N)/{}^{s_j}x]$

Proof. We prove both simultaneously by induction on the structure of M. To start, in the case of ρ', if $\deg(M) = i$ then by Lemma 11, $\deg(M[N/x]) = i$ as well. So

$$\begin{aligned}
\rho'(M[N/x]) &= \bot \to \rho(M[N/x]) \\
&= \bot \to \rho(M)[\rho(N)/x] \\
&= \rho'(M)[\rho(N)/x]
\end{aligned}$$

where the second equality follows from the inductive hypothesis. And if $\deg(M) \neq i$, then

$$\begin{aligned}
\rho'(M[N/x]) &= \rho(M[N/x]) \\
&= \rho(M)[\rho(N)/x] \\
&= \rho'(M)[\rho(N)/x]
\end{aligned}$$

So we can proceed by induction on M for the case of ρ. The cases in which M is a sort or a variable are straightforward. The following are the cases for Π-expressions and λ-expressions.

Π-Expression. If M is of the form $\Pi y^A.\ B$ then

$$\begin{aligned}
\rho((\Pi y^A.\ B)[N/x]) &= \rho(\Pi y^{A[N/x]}.\ B[N/x]) \\
&= \Pi y^{\rho'(A[N/x])}.\ \rho(B[N/x]) \\
&= \Pi y^{\rho'(A)[\rho(N)/x]}.\ \rho(B)[\rho(N)/x] \\
&= \rho(\Pi y^A.\ B)[\rho(N)/x]
\end{aligned}$$

λ-Expression. If M is of the form $\lambda y^A.\ P$ then

$$\begin{aligned}
\rho((\lambda y^A.\ P)[N/x]) &= \rho(\lambda y^{A[N/x]}.\ P[N/x]) \\
&= \lambda y^{\rho(A[N/x])}.\ \rho(P[N/x]) \\
&= \lambda y^{\rho(A)[\rho(N)/x]}.\ \rho(P)[\rho(N)/x] \\
&= \rho(\lambda y^A.\ P)[\rho_i(N)/x]
\end{aligned}$$

The case that M is an application is similar. □

Lemma 17. *(ρ and ρ' preserve β-reductions) For derivable expressions M and N, if $M \to_\beta N$ then $\rho(M) \to_\beta \rho(N)$ and $\rho'(M) \to_\beta \rho'(N)$. Furthermore, if $M =_\beta N$ then $\rho(M) =_\beta \rho(N)$ and $\rho'(M) =_\beta \rho'(N)$.*

Proof. The case of β-equality follows immediately from the case of one-step β-reduction. We prove the one-step case for both ρ and ρ' simultaneously by induction on the structure of the one-step β-reduction relation. It is straightforward to show that the lemma holds in the case of ρ' given that it holds inductively for ρ. So we proceed by induction in the case of ρ.

In the case of a redex,

$$\begin{aligned}\rho((\lambda x^A.\ M)N) &= (\lambda x^{\rho(A)}.\ \rho(M))\rho(N) \\ &\to_\beta \rho(M)[\rho(N)/x] \\ &= \rho(M[N/x])\end{aligned}$$

It is then straightforward to verify that this property holds up to congruence. □

We also note here the important fact that this translation is simple enough not to affect the normalization behavior of the expression after translation.

Fact 1. *For any expression M, if $M \in \mathsf{NF}$, then $\rho(M) \in \mathsf{NF}$. Furthermore, $M \in \mathsf{WN}$ (resp., SN) if and only if $\rho(M) \in \mathsf{WN}$ (resp., SN) and $\mu(M) = \mu(\rho(M))$.*

Next is typability preservation. This ensures type derivations can be used in proving that the term-level translation preserves typability, *e.g.*, for translating the derivation of Π-type judgments for abstraction. The translation targets the super-system of $\lambda\mathcal{S}$ for which s_i is negatable, *i.e.*, where it is possible define types of the form $\bot \to A$ when $\deg(A) = i$.

Definition 16. *For any n-tiered pure type system $\lambda\mathcal{S}$, let $\lambda\mathcal{S}^{\neg i}$ be the n-tiered pure type system with the rules $\mathcal{R}_{\lambda\mathcal{S}^{\neg i}} \triangleq \mathcal{R}_{\lambda\mathcal{S}} \cup \{(s_i, s_i)\}$.*

Lemma 18. *(ρ and ρ' preserve typability) For context Γ and expressions M and A, the following hold.*

1. *If $\Gamma \vdash_{\lambda\mathcal{S}} M : A$ and $\deg M \geq i$ then $\Delta, \rho'(\Gamma) \vdash_{\lambda\mathcal{S}^{\neg}} \rho(M) : \rho(A)$.*
2. *If $\Gamma \vdash_{\lambda\mathcal{S}} A : s_j$ and $\deg A \geq i$ then $\Delta, \rho'(\Gamma) \vdash_{\lambda\mathcal{S}^{\neg}} \rho'(A) : s_j$.*

Proof. We prove both simultaneously by induction on the structure of derivations. For item 2, note that by item 1, we have the inference $\Delta, \rho'(\Gamma) \vdash \rho(A) : s_j$. So if $\deg A \neq i$, we're done, and otherwise we have

$$\frac{\Delta, \rho'(\Gamma) \vdash \bot : s_i \qquad \Delta, \rho'(\Gamma) \vdash \rho(A) : s_i}{\Delta, \rho'(\Gamma) \vdash \bot \to \rho(A) : s_i}$$

Note that this judgment is derivable in $\lambda\mathcal{S}^{\neg}$ by construction.

For item 1, we proceed with each case. The case in which the derivation is a single axiom is straightforward.

Variable Introduction. Suppose the last inference is of the form

$$\frac{\Gamma \vdash A : s_j}{\Gamma, x : A \vdash x : A}$$

where $j \geq i + 1$. Note that $\rho'(A) = \rho(A)$ since $\deg(A) > i$. So by the inductive hypothesis, we have

$$\frac{\Delta, \rho'(\Gamma) \vdash \rho(A) : s_j}{\Delta, \rho'(\Gamma), x : \rho(A) \vdash x : \rho(A)}$$

Weakening. Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s_j}{\Gamma, x : B \vdash M : A}$$

By the inductive hypothesis, $\Delta, \rho'(\Gamma) \vdash \rho(M) : \rho(A)$, so if $\deg(B) \leq i$, then we're done. Otherwise, we have

$$\frac{\Delta, \rho'(\Gamma) \vdash \rho(M) : \rho(A) \qquad \Delta, \rho'(\Gamma) \vdash \rho'(B) : s_j}{\Delta, \rho'(\Gamma), x : \rho'(B) \vdash \rho(M) : \rho(A)}$$

where the right antecedent judgment also follows from the inductive hypothesis.

Product Type Formation. If the last inference is of the form

$$\frac{\Gamma \vdash A : s_j \qquad \Gamma, x : A \vdash B : s_k}{\Gamma \vdash \Pi x^A.\ B : s_k}$$

then by the inductive hypothesis, we have

$$\frac{\Delta, \rho'(\Gamma) \vdash \rho'(A) : s_j \qquad \Delta, \rho'(\Gamma), x : \rho'(A) \vdash \rho(B) : s_k}{\Delta, \rho'(\Gamma) \vdash \Pi x^{\rho'(A)}.\ \rho(B) : s_k}$$

where $k \geq i$. Note we can apply the inductive hypothesis to the left antecedent judgment since $\deg(A) \geq \deg(B)$ by non-dependence and so

$$\deg(A) \geq \deg(\Pi x^A.\ B) \geq i.$$

Abstraction. Suppose the last inference is of the form

$$\frac{\Gamma, x : A \vdash M : B \qquad \Gamma \vdash \Pi x^A.\ B : s_j}{\Gamma \vdash \lambda x^A.\ M : \Pi x^A.\ B}$$

Since $\deg(M) = \deg(\lambda x^A.\ M) \geq i$, we have $\deg(B) > i$, and by non-dependence, we have $\deg(A) \geq \deg(B) > i$. In particular, $\rho'(A) = \rho(A)$. Therefore, we have

$$\frac{\Delta, \rho'(\Gamma), x : \rho(A) \vdash \rho(M) : \rho(B) \qquad \Delta, \rho'(\Gamma) \vdash \Pi x^{\rho(A)}.\ \rho(B) : s_j}{\Delta, \rho'(\Gamma) \vdash \lambda x^{\rho(A)}.\ \rho(M) : \Pi x^{\rho(A)}.\ \rho(B)}$$

Application. Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : \Pi x^A.\ B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B[N/x]}$$

By non-dependence, $\deg A \geq \deg B > \deg M \geq i$ and $\rho'(A) = \rho(A)$. So by the inductive hypothesis we have

$$\frac{\Delta, \rho'(\Gamma) \vdash \rho(M) : \Pi x^{\rho(A)}.\ \rho(B) \qquad \Delta, \rho'(\Gamma) \vdash \rho(N) : \rho(A)}{\Delta, \rho'(\Gamma) \vdash \rho(M)\rho(N) : \rho(B)[\rho(N)/x]}$$

where $\rho(B)[\rho(N)/x] = \rho(B[N/x])$ by Lemma 16.

Conversion. Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s_j}{\Gamma \vdash M : B}$$

where $j \geq i$ and $A =_\beta B$. By the inductive hypothesis, we have

$$\frac{\Delta, \rho'(\Gamma) \vdash \rho(M) : \rho(A) \qquad \Delta, \rho'(\Gamma) \vdash \rho(B) : s_j}{\Delta, \rho'(\Gamma) \vdash \rho(M) : \rho(B)}$$

where $\rho(A) =_\beta \rho(B)$ by Lemma 17. □

2.3.2 Term-Level Translation

Next we define the translation of terms. At a high level, we need to add uninterpreted padding variables to our context for collecting expressions in the bodies of λ-expressions so that the translation maps expressions to I-expressions. There are two issues that need to be addressed.

1. The system $\lambda\mathcal{S}$ may not be expressive enough to carry out the translation.
2. The types of the padding variables may depend on variables in the context, which restricts our ability to generalize or abstract over those variables.

BHS handles these two problems by requiring $\lambda\mathcal{S}$ to satisfy two technical restrictions. The first is *negatability*, which was covered in the previous section, and is necessary to derive the types of translated terms. The second is *cleanliness*, which disallows generalizing or abstracting over variables on which the types of padding variables depend.

Definition 17.

- *A rule (s_i, s_j) is **generalizable** if $(s_{i+1}, s_j) \in \mathcal{R}_{\lambda\mathcal{S}}$.*
- *A rule (s_i, s_j) is **harmless** if neither (s_k, s_j) nor (s_k, s_{j-1}) are in $\mathcal{R}_{\lambda\mathcal{S}}$ when $k > j$.*

*An n-tiered pure type system $\lambda\mathcal{S}$ is **clean** if all its rules are generalizable or harmless.*

BHS uses the notion of cleanliness to characterize systems in which padding variables are definable. These padding variables are polymorphic when possible, in which case they are typed as $\Pi A^{s_j}.\ A \to \bot \to \bot$. But if the rules for defining this type are not available (*i.e.*, the rule (s_j, s_i) is not generalizable) the padding variables are typed as $\rho(A)' \to \bot \to \bot$, with the understanding that $\rho'(A)$ may depend on variables in the context. For an expression M derivable in the context Γ, its translation $\tau(M)$ is derived in a context of the form

$$\Delta, \rho'(\Gamma), \Upsilon$$

where Υ contains padding variables depending on variables in $\Delta, \rho'(\Gamma)$. The challenge is then translating abstractions and generalizations (though the latter is significantly more problematic). To illustrate this, consider that the left antecedent judgment of abstraction is translated roughly to

$$\Delta, \rho'(\Gamma), x : \rho'(A), \Upsilon \vdash \tau(M) : \rho(A)$$

so in order to abstract over the variable x, it must commute with the entire context Υ of padding variables, *i.e.*, it must be that x does not appear free any of the types in Υ. By Lemma 12, the only types which can depend on the variable x are those with higher degree than that of $\rho'(A)$, so cleanliness requires that either

- *(generalizability)* all types of degree larger than $\deg(\rho'(A))$ have polymorphic padding variables, or
- *(harmlessness)* there are no rules for abstracting or generalizing over variables whose types have larger degree than $\deg(\rho'(A))$.

In the next two sections, I present two variants of the term-level translation τ, depending on whether s_i is negatable. In the non-negatable case, τ targets an extension of $\lambda\mathcal{S}$, but is otherwise an application of the ideas in the BHS paper. The deviation is in the new requirement that weak normalization preservation, i-security, and infinite reduction path preservation must be proved explicitly, as we can no longer assume that the target system of τ is weakly normalizing. In the negatable case, τ is a novel variant of the thunkification translation which uses a more complex padding scheme that reduces the structural restrictions on $\lambda\mathcal{S}$, the details of which will be covered in Section 2.3.2.

The Non-Negatable Case

BHS handles the case of irrelevant non-negatable sorts by an argument leveraging the sparse inhabitation of types at irrelevant sorts.

Definition 18. *A sort* s_j *is* ***irrelevant*** *with respect to rules* $\mathcal{R}$ *if there is no sort* s_k *such that* $(s_k, s_j) \in \mathcal{R}$*. A rule* (s_k, s_j) *is said to be* j***-relevant****.*

Lemma 19. *(Barthe et al. [2001], Lemma 5.19) Let* $\lambda\mathcal{S}$ *be a non-dependent, weakly normalizing,* i*-secure tiered pure type system. If* s_i *is irrelevant, then every derivable expression* M *with* $\deg(M) = i - 1$ *is strongly normalizing. That is,* $\lambda\mathcal{S}$ *is* $(i-1)$*-secure.*

The translation below handles non-negatable sorts by translating into a negatable extension of $\lambda\mathcal{S}$. This was also done in the previous section to ensure that translated types were derivable.

Because of Lemma 13, we can work with a slightly weaker notion of cleanliness for this translation.

Definition 19. *A rule* (s_j, s_i) *is* ***weakly harmless*** *if for* $k > j$*, it follows that* $(s_k, s_i) \in \mathcal{R}_{\lambda\mathcal{S}}$ *or* $(s_k, s_{i-1}) \in \mathcal{R}_{\lambda\mathcal{S}}$ *implies* $k \notin \mathcal{D}_{\lambda\mathcal{S}}(j)$*. A non-dependent* n*-tiered pure type system* $\lambda\mathcal{S}$ *is* i***-weakly clean*** *if all* i*-relevant rules are either generalizable or weakly harmless.*

As noted in the introduction to this section, the translation needs access to padding variables so that they can be used to hide terms inside thunks, and these padding functions will be polymorphic when possible; if (s_j, s_i) is generalizable, then it is possible to add a variable pad of type $\Pi A^{s_j}.\ A \to \bot \to \bot$ so that if we need to pad a term N of type $\rho'(A)$ of degree j into a thunkified term M of type $\bot \to \tau(B)$, we can type the term $\lambda \bullet^{\bot} M(\mathsf{pad} A N \bullet)$ as $\bot \to \tau(B)$ as well. If (s_j, s_i) is not generalizable, then the padding variables are typed simply as $\rho'(A) \to \bot \to \bot$ with the caveat that $\rho'(A)$ may have free variables that appear in the context.

Definition 20. *For each index i, define $\Upsilon_{i,M}$ inductively on the structure of M as follows.*

$$\begin{aligned}\Upsilon_{i,s_{i-2}} &\triangleq \varnothing\\ \Upsilon_{i,{}^{s_i}x} &\triangleq \varnothing\\ \Upsilon_{i,\Pi x^A.\ B} &\triangleq \Upsilon_{i,A}, \Upsilon_{i,B}\\ \Upsilon_{i,\lambda x^A.\ M} &\triangleq \mathsf{pad}_x : \alpha_{i,A}, \Upsilon_{i,M}\\ \Upsilon_{i,MN} &\triangleq \Upsilon_{i,M}\end{aligned}$$

where

$$\alpha_{i,A} = \begin{cases}\Pi t^{s_{\deg(A)}}.\ t \to \bot_i \to \bot_i & (\deg(A), i) \text{ is generalizable}\\ \rho'_i(A) \to \bot_i \to \bot_i & (\deg(A), i) \text{ is weakly harmless}\end{cases}$$

I will write $\langle M, N\rangle_{\rho(A)}$ for both $\mathsf{pad}_x \rho(A) M N$ when $(\deg(A), i)$ is generalizable and $\mathsf{pad}_x M N$ when $(\deg(A), i)$ is weakly harmless, if there is no fear of ambiguity.

Fact 2. *For any derivable expression M, the context Δ, Υ_M is well-formed in $\lambda\mathcal{S}^{\neg}$.*

The context of padding variables appears *after* the translation of a given context, as it may depend on some the declarations in it, so to translate abstractions and generalizations, we need some declarations to commute with the context of padding variables. The following ensures this is possible.

Lemma 20. *If (s_j, s_i) or (s_j, s_{i-1}) are in $\mathcal{R}_{\lambda\mathcal{S}}$ then no variables (other than $\bot$) of degree $j-1$ (i.e., of the form ${}^{s_j}x$) can appear free in Υ_M for any derivable expression M where* $\deg(M) = i - 1$.

Proof. By induction on the structure of M. The only interesting case is if M is of the form $\lambda x^A.\ N$ where $(\deg(A), i)$ is weakly harmless. If, for some k, the variable ${}^{s_k}x$ appears free in $\rho'(A)$, then it must also appear free in A (this is straightforward to check), and so by

Lemma 13, it must be that $k > \deg(A)$ and $k \in \mathcal{D}_{\lambda\mathcal{S}}(\deg(A))$. But by weak harmlessness, $j \notin \mathcal{D}_{\lambda\mathcal{S}}(\deg(A))$, so must be that $j \neq k$. □

As for the translation itself, it is quite natural. The core of the definition is in the case of λ-expressions, where the padding variables are used to ensure our translation maps expressions to I-expressions.

Definition 21. *Define the functions*

$$\tau_i : \mathsf{T}_{i-1} \to \mathsf{T}_{i-1} \qquad \textit{and} \qquad \tau_i' : \mathsf{T}_{i-1} \to \mathsf{T}_{i-1}$$

simultaneously as follows.

$$\begin{aligned}
\tau_i(s_{i-2}) &\triangleq \bullet_i \\
\tau_i({}^{s_i}x) &\triangleq {}^{s_i}x\bullet_i \\
\tau_i(\Pi x^A.\ B) &\triangleq \begin{cases} \tau_i(A) \to \tau_i(B) & \deg(A) = i-1 \\ \Pi x^{\rho_i'(A)}.\ \tau_i(B) & \textit{otherwise} \end{cases} \\
\tau_i(\lambda x^A.\ M) &\triangleq \lambda x^{\rho_i'(A)}.\ \tau_i'(M)\langle x, \bullet_i\rangle_{\rho_i'(A)} \\
\tau_i(MN) &\triangleq \tau_i(M)\rho_i(N) \\
\tau_i'(M) &\triangleq \lambda \bullet_i^{\perp_i}.\ \tau_i(M)
\end{aligned}$$

where $j \geq i-1$ and $\bullet_i$ is a distinguished variable.

This translation must first be shown to be well-defined. In particular, it must be that if MN is derivable, then $\deg(N) \geq i$, so that ρ may be applied. This is the first place where we use non-negatability. By generation, M must be be given a Π-type $\Pi x^A.\ B$ where $\deg(B) = i$ by assumption and $\deg(A) \geq i$ by non-dependence. By non-negatability, it must in fact be that $\deg(A) \geq i+1$, and so $\deg(N) \geq i$.

Note that, for sorts, $\bot_i = s_{i-1}$, so $\tau(s_{i-1})$ will be given the correct type. This trick is used to ensure that the translation maps to I-expressions even for abstractions over thunks.

Fact 3. *For any derivable expression M such that $\deg(M) = i - 1$, the variable $\bullet$ appears free in $\tau(M)$.*

Now we verify that this translation preserves typability. This is fairly mechanical. Again, recall that we are checking typability *in a stronger system*.

Lemma 21. *(τ and τ' preserve typability) For any context Γ and expressions M and A, if $\Gamma \vdash_{\lambda\mathcal{S}} M : A$ and $\Gamma \vdash_{\lambda\mathcal{S}} A : s_i$, then*

1. $\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_M \vdash_{\lambda\mathcal{S}^\neg} \tau(M) : \rho(A)$
2. $\Delta, \rho'(\Gamma), \Upsilon_M \vdash_{\lambda\mathcal{S}^\neg} \tau'(M) : \rho'(A)$

Proof. We prove both simultaneously by induction on the structure of derivations. For item 2, suppose that

$$\Delta, \bullet : \bot, \Upsilon_M, \rho'(\Gamma), \vdash \tau(M) : \rho(A)$$

By typability preservation of ρ (Lemma 18) and thinning (Lemma 4) we can derive

$$\Delta, \rho'(\Gamma), \Upsilon_M \vdash \bot \to \rho(A) : s_i$$

and by permutation (Lemma 5) we can derive

$$\Delta, \rho'(\Gamma), \Upsilon_M, \bullet : \bot \vdash \tau(M) : \rho(A)$$

which means by abstraction we can derive

$$\Delta, \Upsilon, \rho'(\Gamma) \vdash \lambda \bullet^{\bot}.\ \tau(M) : \bot \to \rho(A)$$

For item 1, we consider each case.

Axiom. If the derivation is of the form

$$\vdash s_{i-2} : s_{i-1}$$

then of course

$$\Delta, \bullet_i : \bot_i \vdash \bullet_i : \bot_i$$

Recall that $\bot_i = s_{i-1} = \rho(s_{i-1})$.

Variable Introduction. Suppose the last derivation is of the form

$$\frac{\Gamma \vdash A : s_i}{\Gamma, x : A \vdash x : A}$$

By ρ typability preservation and thinning, we can derive

$$\frac{\Delta, \bullet : \bot, \rho'(\Gamma) \vdash \rho'(A) : s_i}{\Delta, \bullet : \bot, \rho'(\Gamma), x : \rho'(A) \vdash x : \rho'(A)}$$

and then we can use weakening and application to derive

$$\Delta, \bullet : \bot, \rho'(\Gamma), x : \rho'(A) \vdash x\bullet : \rho(A)$$

Weakening. Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s_j}{\Gamma, x : B \vdash M : A}$$

If $j < i$, then we have

$$\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_M \vdash \tau(M) : \rho(A)$$

directly by the inductive hypothesis. Otherwise, B is in the domain of ρ' and by the inductive hypothesis, ρ typability preservation, and thinning, we have

$$\frac{\begin{array}{c}\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_M \vdash \tau(M) : \rho(A) \\ \Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_M \vdash \rho'(B) : s_j\end{array}}{\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_M, x : \rho'(B) \vdash \tau(M) : \rho(A)}$$

Since none of the padding variables in Υ_M appear in $\rho'(B)$ it follows by permutation that we can also derive

$$\Delta, \bullet : \bot, \rho'(\Gamma), x : \rho'(B), \Upsilon_M \vdash \tau(M) : \rho(A)$$

<u>Product Type Formation.</u> Suppose the last inference is of the form

$$\frac{\Gamma \vdash A : s_j \qquad \Gamma, {}^{s_j}x : A \vdash B : s_{i-1}}{\Gamma \vdash \Pi x^A.\ B : s_{i-1}}$$

If $j = i - 1$, then by the inductive hypothesis, and thinning we have

$$\frac{\begin{array}{c}\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_A, \Upsilon_B \vdash \tau(A) : s_{i-1} \\ \Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_A, \Upsilon_B \vdash \tau(B) : s_{i-1}\end{array}}{\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_A, \Upsilon_B \vdash \tau(A) \to \tau(B) : s_{i-1}}$$

Otherwise, A is in the domain of ρ' and the inductive hypothesis gives us

$$\Delta, \bullet : \bot, \rho'(\Gamma), x : \rho'(A), \Upsilon_B \vdash \tau(B) : s_{i-1}$$

By Lemma 20, x does not appear free in Υ_B, so by permutation, ρ typability preservation and thinning, we can derive

$$\frac{\begin{array}{c}\Delta, \bullet : \bot \rho'(\Gamma), \Upsilon_A, \Upsilon_B \vdash \rho'(A) : s_j \\ \Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_A, \Upsilon_B, {}^{s_j}x : \rho'(A) \vdash \tau(B) : s_{i-1}\end{array}}{\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_A, \Upsilon_B \vdash \Pi x^{\rho'(A)}.\ \tau(B) : s_{i-1}}$$

<u>Abstraction.</u> Suppose the last inference is of the form

$$\frac{\Gamma, {}^{s_j}x : A \vdash M : B \qquad \Gamma \vdash \Pi x^A.\ B : s_i}{\Gamma \vdash \lambda x^A.\ M : \Pi x^A.\ B}$$

By the inductive hypothesis,

$$\Delta, \rho'(\Gamma), x : \rho'(A), \Upsilon_M \vdash \tau'(M) : \bot \to \rho(B)$$

and by Lemma 20 and permutation, in fact we have

$$\Delta, \rho'(\Gamma), \Upsilon_M, x : \rho'(A) \vdash \tau'(M) : \bot \to \rho(B)$$

By assumption, (s_j, s_i) is generalizable or weakly harmless. In the first case, we have

$$\Delta \vdash \Pi A^{s_j}.\ A \to \bot \to \bot : s_i$$

Otherwise, by ρ typability preservation and a couple additional steps (including a use of generation), we can derive

$$\Delta, \rho'(\Gamma) \vdash \rho'(A) \to \bot \to \bot : s_i$$

so in both cases, we can apply thinning to derive.

$$\Delta, \rho'(\Gamma), \Upsilon_{\lambda x^A.\ M}, x : \rho'(A) \vdash \tau'(M) : \bot \to \rho(B)$$

Therefore, in a few steps we can derive

$$\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_{\lambda x^A.\ M}, x : \rho'(A) \vdash \langle x, \bullet \rangle_{\rho'(A)} : \bot$$

which can be used with application to derive

$$\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_{\lambda x^A.\ M}, x : \rho'(A) \vdash \tau'(M)\langle x, \bullet \rangle_{\rho'(A)} : \rho(B)$$

Finally, this can be used with abstraction to derive

$$\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_{\lambda x^A.\ M} \vdash \lambda x^{\rho'(A)}.\ \tau'(M)\langle x, \bullet \rangle_{\rho'(A)} : \Pi x^{\rho'(A)}.\ \rho(B)$$

Application. Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : \Pi x^A.\ B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B[N/x]}$$

As noted above, by non-negatability, it must be that $\deg(N) \geq i$, which implies N is in the domain of ρ, so with additional thinning we have

$$\frac{\begin{array}{c}\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_M, \vdash \tau(M) : \Pi x^{\rho(A)}.\ \rho(B) \\ \Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_M \vdash \rho(N) : \rho(A)\end{array}}{\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_M \vdash \tau(M)\rho(N) : \rho(B)[\rho(N)/x]}$$

and $\rho(B)[\rho(N)/x] = \rho(B[N/x])$ by Lemma 16.

<u>Conversion.</u> Suppose the last inference is of the form.

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s_j}{\Gamma \vdash M : B}$$

where $A =_\beta B$. By the inductive hypothesis, ρ-typability preservation, thinning, and ρ β-preservation (Lemma 17), we have

$$\frac{\begin{array}{c}\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_M \vdash \tau(M) : \rho(A) \\ \Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_M \vdash \rho(B) : s_j\end{array}}{\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon_M \vdash \tau(M) : \rho(B)}$$

□

The focus of the remainder of the section is to prove the key lemmas below which state that

- (Lemma 32) τ preserves weak normalization *of terms* (but not necessarily of the system as a whole).
- (Lemma 26 i-security is preserved from $\lambda\mathcal{S}$ to $\lambda\mathcal{S}^\neg$, and hence, the conservation lemma (Lemma 15) holds for to $\lambda\mathcal{S}^\neg$.
- (Lemma 34) τ preserves infinite reduction sequences.

These encompass the three steps of the outline given at the beginning of the section. I start with the second result since it fairly general and useful in proving the remaining to lemmas.

i-security Preservation. The basic problem is the following: $\lambda\mathcal{S}$ and $\lambda\mathcal{S}^\neg$ derive the same expressions of degree greater than i, but there is a small class of degree i expressions that are derivable with the introduction of the new rule (s_i, s_i), namely function types of the form

$A \to B$ where $\deg(A) = \deg(B) = i$ (there can be no dependence because all variables of B must have types of degree greater than i). Without loss of generality we may annotate these types as $A \to^i B$.

It is necessary to prove that the introduction of these types does not allow for non-strongly-normalizing expressions. Morally speaking, in degree i expressions, these function types are equivalent to pair types; any substitution that happens in Π-types only occurs to the right of the turnstile, when it is treated as a type. So the real challenge is to prove that pair types for degree i types do not allow for non-strongly-normalizing expressions. I emphasize here that this only refers to *types*. We don't need to worry about how these types are inhabited, since such an expression would be of degree $i - 1$.

Let M be an expression of degree i derivable in $\lambda\mathcal{S}^{\neg}$ and suppose that M has an infinite reduction sequence. The rough idea is to think of $A \to^i B$ appearing in M as a non-deterministic choice between A and B in the derivation of M. Considering all non-deterministic choices, we construct a set of *projections* of M, which are derivable without an uses of $\to^i$, and hence derivable in $\lambda\mathcal{S}$. It will then be possible to track the development of reductions in these projections alongside the infinite reduction sequence of M and show that there is always some projection which can make progress for every reduction in the sequence. Since the set of candidates for M is finite, some candidate expression must make infinite progress.

Definition 22. *The set of* ***projections*** *of a derivable expression M of $\lambda\mathcal{S}^{\neg}$ is defined in-*

ductively as follows.

$$\begin{aligned}
\mathsf{Proj}(s_j) &\triangleq \{s_j\} \\
\mathsf{Proj}(x) &\triangleq \{x\} \\
\mathsf{Proj}(\Pi x^A.\ B) &\triangleq \{\Pi x^{A'}.\ B' \mid A' \in \mathsf{Proj}(A) \text{ and } B' \in \mathsf{Proj}(B)\} \\
\mathsf{Proj}(\lambda x^A.\ M) &\triangleq \{\lambda x^{A'}.\ M' \mid A' \in \mathsf{Proj}(A) \text{ and } M' \in \mathsf{Proj}(M)\} \\
\mathsf{Proj}(MN) &\triangleq \{M'N' \mid M' \in \mathsf{Proj}(M) \text{ and } N' \in \mathsf{Proj}(N)\} \\
\mathsf{Proj}(A \to^i B) &\triangleq \mathsf{Proj}(A) \cup \mathsf{Proj}(B)
\end{aligned}$$

Lemma 22. *If M is derivable in $\lambda\mathcal{S}^{\neg}$, then every projection of M is derivable in $\lambda\mathcal{S}$. Furthermore, if $\deg(M) = i$, then every projection is of degree i as well.*

Lemma 23. *For derivable expression M and N such that $\deg(M) = i$,*

$$\mathsf{Proj}(M[N/x]) = \{M'[N'/x] \mid M' \in \mathsf{Proj}(M) \text{ and } N' \in \mathsf{Proj}(N)\}$$

The proofs of the above lemmas are straighforward. Unfortnately, projections do not behave very well with respect to β-reduction. There are two issues to address.

- There may be redexes in an expression that are not represented in its projections. In the simplest case, a redex appearing in the left hand side of the expression $A \to^i B$ does not have an analogous redex in any of the projections of B appearing in $\mathsf{Proj}(A \to^i B)$.

- Even when every projection of M is reduced along the analogous redex when it exists, the resulting set of projections may not be a full set of projections of the reduct of M. Reducing any redex which creates multiple copies of its argument will yield an expression M' whose projections contains expressions that are not the result of reducing projections of M.

The first problem is addressed by inductively defining a β-reduction operation for sets of projections, which may act as the identity on some projections. The second problem is addressed by defining the notion of *coverage*; even if a set of projections does not contain all of $\mathsf{Proj}(M)$, there may be sufficiently many of them that they capture enough information about M. Roughly speaking, a set of projections covers M all non-deterministic branches in the expression tree are considered, even if not all combinations are considered.

Definition 23. *Let M and N be a derivable expressions in $\lambda\mathcal{S}^{\neg}$ such that $M \to_\beta N$ along the redex R, and let M' be a projection of M. Define $\beta_R^M(M')$ by induction on the structure of M as follows. Note that M cannot be a sort or a variable.*

$$\beta_R^{\Pi x^A.\ B}(\Pi x^{A'}.\ B') \triangleq \begin{cases} \Pi x^{\beta_R^A(A')}.\ B' & R \subset A \\ \Pi x^{A'}.\ \beta_R^B(B') & R \subset B \end{cases}$$

$$\beta_R^{\lambda x^A.\ M}(\lambda x^{A'}.\ M') \triangleq \begin{cases} \lambda x^{\beta_R^A(A')}.\ M' & R \subset A \\ \lambda x^{A'}.\ \beta_R^M(M') & R \subset M \end{cases}$$

$$\beta_R^{MN}(M'N') \triangleq \begin{cases} \beta_R^M(M')N' & R \subset M \\ M'\beta_R^N(N') & R \subset N \\ P[Q/x] & R = MN = (\lambda x^A.\ P)Q \end{cases}$$

$$\beta_R^{A \to^i B}(M) \triangleq \begin{cases} \beta_R^A(M) & M \in \mathsf{Proj}(A), R \subset A \\ \beta_R^B(M) & M \in \mathsf{Proj}(B), R \subset B \\ M & \textit{otherwise} \end{cases}$$

For a subset $\mathcal{P}$ of $\mathsf{Proj}(M)$, define $\beta_R^M(\mathcal{P}) \triangleq \{\beta_R^M(P) \mid P \in \mathcal{P}\}$.

Lemma 24. *For derivable expressions M and N such that $M \to_\beta N$ along R, if $\mathcal{P} \subset \mathsf{Proj}(M)$, then $\beta_R^M(\mathcal{P}) \subset \mathsf{Proj}(N)$.*

This lemma is proved by induction on the structural definition of a redex and then using Lemma 23 in the case of redex. Next, the notion of coverage.

Definition 24. *For an expression of the form $\Pi x^A.\ B$ and a subset $\mathcal{P}$ of $\mathsf{Proj}(\Pi x^A.\ B)$ let $\mathcal{P}_A$ denote the set $\{A' \mid \Pi x^{A'}.\ B' \in \mathcal{P}\}$ and let $\mathcal{P}_B$ denote the set $\{B' \mid \Pi x^{A'}.\ B' \in \mathcal{P}\}$. I will use the analogous notation for λ-expressions and applications. In the case of $\rightarrow^i$, let $\mathcal{P}_A$ denote the set $\mathcal{P} \cap \mathsf{Proj}(A)$ and let $\mathcal{P}_B$ denote the set $\mathcal{P} \cap \mathsf{Proj}(B)$. For any expression M, a set of projections $\mathcal{P}$ from $\mathsf{Proj}(M)$ **covers** M if the following hold.*

- *$\mathcal{P}$ covers s_j if $\mathcal{P} = \{s_j\}$.*
- *$\mathcal{P}$ covers ${}^{s_j}x$ if $\mathcal{P} = \{{}^{s_j}x\}$.*
- *$\mathcal{P}$ covers $\Pi x^A.\ B$ if $\mathcal{P}_A$ covers A and $\mathcal{P}_B$ covers B.*
- *$\mathcal{P}$ covers $\lambda x^A.\ M$ if $\mathcal{P}_A$ covers A and $\mathcal{P}_M$ covers M.*
- *$\mathcal{P}$ covers MN if $\mathcal{P}_M$ covers M and $\mathcal{P}_N$ covers N.*
- *$\mathcal{P}$ covers $A \rightarrow^i B$ if $\mathcal{P}_A$ covers A and $\mathcal{P}_B$ covers B.*

The fundamental lemma of coverage is that it is preserved by β-reduction.

Lemma 25. *Let M and N be expressions such that $M \rightarrow_\beta N$ along the redex R and let $\mathcal{P}$ be a set of projections of M. If $\mathcal{P}$ covers M then $\beta_R^M(\mathcal{P})$ covers N. Furthermore, there is some projection P of $\mathcal{P}$ such that $P \rightarrow_\beta \beta_R^M(P)$ (in particular, $\beta_R^M(P) \neq P$).*

Proof. (Sketch) This follows by induction on notion of coverage, using Lemma 23 in the case of a redex. In this process, it is possible to build inductively a witness to the fact that some element in $\mathcal{P}$ is β-reduced. □

Lemma 26. *If $\lambda\mathcal{S}$ is i-secure, then $\lambda\mathcal{S}^\neg$ is i-secure.*

Proof. Suppose that $\lambda\mathcal{S}^\neg$ is not i-secure. Let M be an expression such that $\deg(M) \geq i$ and $M \notin \mathsf{SN}$. Since $\lambda\mathcal{S}$ and $\lambda\mathcal{S}^\neg$ derive the same expressions of degree greater than i, it must be that $\deg(M) = i$. Given an infinite reduction sequence starting at M, Lemma 25 implies there is an infinite sequences of β reductions on sets of projections starting at $\mathsf{Proj}(M)$ (which trivially covers M). Since $\mathsf{Proj}(M)$ is finite, there some expression in $\mathsf{Proj}(M)$ which has an infinite reduction sequence. Since all expression in $\mathsf{Proj}(M)$ are derivable in $\lambda\mathcal{S}$ and are degree i, it follows that $\lambda\mathcal{S}$ is not i-secure. □

Weak Normalization Preservation. First, a substitution commutation lemma akin to Lemma 16 but for τ.

Lemma 27. *(τ commutes with substitution) For variable ${}^{s_k}x$ and derivable expressions M and N where* $\deg(M) = i - 1$ *and* $\deg(N) \geq i$*,*

$$\tau(M[N/{}^{s_k}x]) = \tau(M)[\rho(N)/{}^{s_k}x]$$

Proof. By induction on the structure of M. The cases in which M is a sort or a variable are straightforward.

Π-Expression. Suppose M is of the form $\Pi y^A.\ B$. If $\deg(A) = i - 1$, then

$$\begin{aligned}
\tau((\Pi y^A.\ B)[N/x]) &= \tau(A[N/x]) \to \tau(B[N/x]) \\
&= \tau(A)[\rho(N)/x] \to \tau(B)[\rho(N)/x] \\
&= \tau(\Pi y^A.\ B)[\rho(N)/x]
\end{aligned}$$

Otherwise,

$$\begin{aligned}\tau((\Pi y^{A}.\ B)[N/x]) &= \Pi y^{\rho'(A[N/x])}.\ \tau(B[N/x]) \\ &= \Pi y^{\rho'(A)[\rho(N)/x]}.\ \tau(B)[\rho(N)/x] \\ &= \Pi y^{\rho'(A)[\rho(N)/x]}.\ \tau(B)[\rho(N)/x] \\ &= \tau(\Pi y^{A}.\ B)[\rho(N)/x]\end{aligned}$$

The cases that M is a λ-terms or an application are similar. □

Unfortunately, unlike ρ, the translation τ does not preserve β-reductions. This is by design, as it maintains information about previously performed reductions in the padded expressions. So, to aid the proof below, we use an auxiliary translation θ, which is simply τ without padding. This translation *does* preserve β-reductions, and will act as a convenient intermediate translation.

Definition 25. *Define the functions*

$$\theta_i : \mathsf{T}_{i-1} \to \mathsf{T}_{i-1} \qquad \textit{and} \qquad \theta'_i : \mathsf{T}_{i-1} \to \mathsf{T}_{i-1}$$

simultaneously as follows.

$$\theta_i(s_{i-2}) \triangleq \bullet_i$$

$$\theta_i({}^{s_i}x) \triangleq {}^{s_i}x\bullet_i$$

$$\theta_i(\Pi x^A.\ B) \triangleq \begin{cases} \theta_i(A) \to \theta_i(B) & \deg(A) = i-1 \\ \Pi x^{\rho'_i(A)}.\ \theta_i(B) & \textit{otherwise} \end{cases}$$

$$\theta_i(\lambda x^A.\ M) \triangleq \lambda x^{\rho'_i(A)}.\ \theta'_i(M)\bullet$$

$$\theta_i(MN) \triangleq \theta_i(M)\rho_i(N)$$

$$\theta'_i(M) \triangleq \lambda \bullet_i{}^{\bot_i}.\ \theta_i(M)$$

where $j \geq i-1$ *and* $\bullet_i$ *is a distinguished variable.*

Again, the definition of θ is exactly the same as that of τ, except in the case of λ-expressions, where no padding occurs. In particular, θ does not map expressions to I-expression. But it does have the same sort of substitution commutation and β-reduction preservation lemmas as ρ, and it can further be proved that θ preserves weak normalization.

Lemma 28. *(θ commutes with substitution) For variable ${}^{s_k}x$ and derivable expressions M and N where* $\deg(M) = i-1$ *and* $\deg(N) \geq i$*,*

$$\theta(M[N/{}^{s_k}x]) = \theta(M)[\rho(N)/{}^{s_k}x]$$

The proof is nearly identical to that of Lemma 27 above.

Lemma 29. *For derivable expression M and N of degree $i-1$, the following hold.*

1. *If $M \to_\beta N$, then $\theta(M) \twoheadrightarrow_\beta \theta(N)$.*

2. $\theta(\mathsf{NF}(M)) \twoheadrightarrow_\beta \mathsf{NF}(\theta(M))$.

Proof. The proof of item 1 is straightforward, modulo Lemma 28. For the proof of item 2 it suffices to note that performing a reduction of the form $\theta'(N)\bullet \to_\beta \theta(N)$ in $\theta(M)$ does not produce any new redexes if $M \in \mathsf{NF}$. $\square$

The translations τ and θ are then connected via an erasure function.

Definition 26. *Define the function* $|\cdot| : \mathsf{T}_{i-1} \to \mathsf{T}_{i-1}$ *which erases padding expressions shallowly, i.e.,*

$$
\begin{aligned}
|\langle P, Q\rangle_A| &\triangleq \bullet_i \\
|s_{i-2}| &\triangleq s_{i-2} \\
|{}^{s_i}x| &\triangleq {}^{s_i}x \\
|\Pi x^A.\ B| &\triangleq \Pi x^{|A|}.\ |B| \\
|\lambda x^A.\ B| &\triangleq \lambda x^{|A|}.\ |B| \\
|MN| &\triangleq |M||N|
\end{aligned}
$$

Fact 4. $|\tau(M)| = \theta(M)$ *for any expression* M*.*

Naturally, this erasure function also commutes with substitution.

Fact 5. $|M[N/x]| = |M|[|N|/x]$ *for variable* ${}^{s_k}x$ *and expressions* M *and* N *where* $\deg(M) = i - 1$ *and* $\deg(N) = k - 1$*.*

The key idea of the following lemma is to build a reduction sequence starting at $\tau(M)$ which follows a reduction sequence starting at M in lockstep, and then argue that the padded information in the reductions of $\tau(M)$ is sufficiently simple so as not to induce any potentially infinite reduction sequences. The intermediate translation θ is used as a lens to view reduced forms of $\tau(M)$ without the additional padding information.

A padded expression is sufficiently simple it's only non-thunk expressions are of degree at least i. Since $\lambda\mathcal{S}$, and hence $\lambda\mathcal{S}^{\neg}$, is i-secure, the padded expressions must be strongly normalizing. Note that $\bullet$, and $\langle M, N\rangle_A$ are both degree $i-1$, so this is not immediate, but it is not difficult to show in the case that $\lambda\mathcal{S}$ is not i-negatable.

Definition 27. *An expression M is i-**padded** if $\langle P, Q\rangle_A \subset M$ implies it is of the from*

$$\langle P_1, \langle P_2, \ldots, \langle P_k, \bullet\rangle_{A_k} \cdots\rangle_{A_2}\rangle_{A_1}$$

*where $P_1, \ldots, P_k$ (and hence, $A_1, \ldots, A_k$) are degree at least i. I will call such an expression an i-**padding term**.*

The above idea is captured by the following commutative diagram. Note that there are two different kinds of arrows in this diagram; horizontal arrows are for β-reductions and vertical arrows are for translations. In essense, the lemma says that, although τ does not preserve β-reduction in the same way that θ does, it can be made to preserve it up to ρ-padding, which is sufficient since $\tau(M)$ is ρ-padded by design.

Lemma 30. *Let M, M' and N be derivable expressions such that $M \to_\beta M'$ and $|N| = \theta(M)$ and N is i-padded. Then there is a derivable expression N' such that N' is i-padded and the following diagram commutes.*

$$\begin{array}{ccc}
M & \xrightarrow{\beta} & M' \\
\downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\
\theta(M) & \xrightarrow{\beta} & \theta(M') \\
\uparrow{\scriptstyle|\cdot|} & & \uparrow{\scriptstyle|\cdot|} \\
N & \overset{\beta}{\twoheadrightarrow} & N'
\end{array}$$

Proof. Let C be a single-holed expression such that $M = C\langle(\lambda x^A.\ P)Q\rangle)$ and $M' = C\langle P[Q/x]\rangle$, *i.e.*, $(\lambda x^A.\ P)Q$ is the redex along which M is reduced to M'. By definition of

θ, there is a single-holed expression C' such that

$$\theta(M) = C'\langle(\lambda x^{\rho'(A)}.\ (\lambda \bullet^{\perp}.\ \theta(P))\bullet)\rho(Q)\rangle$$

By definition of $|\cdot|$, there is some context C'' such that

$$N = C''\langle(\lambda x^{\rho'(A)}.\ \lambda \bullet^{\perp}.\ P')Z)\rho(Q)\rangle$$

where $|P'| = \theta(P)$ and $|Z| = \bullet$. Note that, since $|\cdot|$ only removes padding information, we know that this redex must appear in N (*i.e.*, it is not erased by $|\cdot|$) and that it does not affect ρ-translated expressions. Since N is i-padded, it must be that

$$Z = \langle T_1, \langle T_2, \ldots, \langle T_k, \bullet\rangle_{A_k} \cdots\rangle_{A_2}\rangle_{A_1}$$

and Z is an i-padding term. Then

$$\begin{aligned} C''\langle(\lambda x^{\rho'(A)}.\ (\lambda \bullet^{\perp}.\ P')Z)\rho(Q)\rangle &\rightarrow_\beta C''\langle(\lambda \bullet^{\perp}.\ P'[\rho(Q)/x])Z[\rho(Q)/x]\rangle \\ &\rightarrow_\beta C''\langle P'[\rho(Q)/x][Z[\rho(Q)/x]/\bullet]\rangle \end{aligned}$$

Take N' to be this last expression, which is i-padded because degree is preserved by substitution (which handles the first reduction) and substitution $\bullet$ in an i-padding expression with an i-padding expression clearly preserving i-padding. Finally,

$$\begin{aligned} |C''\langle P'[\rho(Q)/x])[Z[\rho(Q)/x]/\bullet]\rangle| &= C'\langle\theta(P[Q/x])[|Z[\rho(Q)/x]|/\bullet]\rangle \\ &= C'\langle\theta(P[Q/x])[\bullet/\bullet]\rangle \\ &= C'\langle\theta(P[Q/x])\rangle \\ &= \theta(C\langle P[Q/x]\rangle) \end{aligned}$$

$$
\begin{aligned}
|C''\langle P'[\rho(Q)/x]\rangle| &= C'\langle |P'[\rho(Q)/x]| \rangle \\
&= C'\langle \theta(P)[\rho(Q)/x] \rangle \\
&= C'\langle \theta(P[Q/x]) \rangle \\
&= \theta(C\langle P[Q/x] \rangle)
\end{aligned}
$$

$\square$

Lemma 31. *Let M and M' be derivable expressions such that $M \twoheadrightarrow_\beta M'$. Then there is an expression i-padded N such that the following diagram commutes.*

$$
\begin{array}{ccc}
M & \xrightarrow{\ \beta\ } \!\!\!\!\rightarrow & M' \\
\big\downarrow \theta & & \big\downarrow \theta \\
\theta(M) & \xrightarrow{\ \beta\ } \!\!\!\!\rightarrow & \theta(M') \\
\big\uparrow |\cdot| & & \big\uparrow |\cdot| \\
\tau(M) & \overset{\beta}{\cdots\cdots\twoheadrightarrow} & N
\end{array}
$$

Proof. By a diagram chase with Lemma 30. Note that $\tau(M)$ is i-padded by construction. $\square$

All together, this gives us the next key lemma.

Lemma 32. *For any derivable expression M such that* $\deg(M) = i - 1$, *if $M \in \mathsf{WN}$ then $\tau(M) \in \mathsf{WN}$.*

Proof. Consider a reduction sequence from M to $\mathsf{NF}(M)$. By Lemma 31, there is an reduct N of $\tau(M)$ such that $|N| = \theta(\mathsf{NF}(M))$. Following the reduction sequence $\theta(\mathsf{NF}(M)) \twoheadrightarrow_{\mathsf{NF}} (\theta(M))$ given by Lemma 29 analogously in N yields and expression N' such that $|N'| = \mathsf{NF}(\theta(M))$. Thus, the only sub-expressions of N' which have redexes are i-padding expressions, which are strongly normalizing by Lemma 26. $\square$

Infinite Reduction Sequence Preservation. Finally, we prove $\mu(M) \leq \mu(\tau(M))$. Note that I've already stated the upper bound for ρ (Fact 1). We take advantage of a standard structural lemma which is also proved by BHS (Lemma 5.7).

Lemma 33. *If M is derivable then M is a sort, a Π-expressions, or of the form*

$$NM_1 \dots M_k$$

where $k \geq 0$ and N is a variable or a λ-expression.

Lemma 34. *For any derivable expression M with* $\deg(M) = i - 1$,

$$\mu(M) \leq \mu(\tau(M)) = \mu(\tau'(M))$$

Proof. By induction on $\mu(\tau(M))$ and the structure of M, lexicographically ordered. The cases in which M is a sort, a variable, or a Π-expression are straightforward.

λ-Expression. Suppose M is of the form $\lambda x^A.\ N$. By Fact 1, $\mu(A) \leq \mu(\rho'(A))$ and by the inductive hypothesis, $\mu(N) \leq \mu(\tau(N)) = \mu(\tau'(N))$, so

$$\begin{aligned}\mu(\lambda x^A.\ N) &= \mu(A) + \mu(N) \\ &\leq \mu(\rho'(A)) + \mu(\tau(N)) = \mu(\rho'(A)) + \mu(\tau'(N))\end{aligned}$$

And since $\mu(\tau'(N)) \leq \mu(\tau'(N)\langle x, \bullet\rangle_{\rho'(A)})$ we also have that $\mu(\lambda x^A.\ N) \leq \mu(\tau(\lambda x^A.\ N))$.

Application. We consider two cases. First suppose that M is of the form $xN_1 \dots, N_k$ where $k \geq 1$. Then

$$\begin{aligned}\mu(xN_1 \dots N_k) &= \sum_{j=1}^{k} \mu(N_j) \\ &\leq \sum_{j=1}^{k} \mu(\rho(N_j)) \\ &= \mu(\tau(xN_1 \dots N_k)) = \mu(\tau'(xN_1 \dots N_k))\end{aligned}$$

Otherwise, M is of the form $(\lambda x^A.\ P)N_1 \dots N_k$ with $\deg(P) = i - 1$. Consider the reduction sequence

$$
\begin{aligned}
&\tau((\lambda x^A.\ P)N_1 \dots N_k) \\
&= (\lambda x^{\rho'(A)}.\ \tau'(P)\langle x, \bullet\rangle_{\rho'(A)})\rho(N_1), \dots \rho(N_k) \\
&\twoheadrightarrow_\beta (\lambda x^{\mathsf{NF}(\rho'(A))}.\ \tau'(P)\langle x, \bullet\rangle_{\rho'(A)})\rho(N_1), \dots \rho(N_k) \\
&\rightarrow_\beta \tau'(P)[\rho(N_1)/x]\langle \rho(N_1), \bullet\rangle_{\rho'(A)}\rho(N_2) \dots \rho(N_k) \\
&= \tau'(P[N_1/x])\langle \rho(N_1), \bullet\rangle_{\rho'(A)}\rho(N_2) \dots \rho(N_k) \\
&\twoheadrightarrow_\beta \tau'(P[N_1/x])\langle \mathsf{NF}(\rho(N_1)), \bullet\rangle_{\rho'(A)}\rho(N_2) \dots \rho(N_k) \\
&\rightarrow_\beta \tau(P[N_1/x])[\langle \mathsf{NF}(\rho(N_1)), \bullet\rangle_{\rho'(A)}/\bullet]\rho(N_2) \dots \rho(N_k)
\end{aligned}
$$

where the above normal forms are guaranteed to exist by the assumption of i-security and Lemma 26. Note that the fifth line follows from Lemma 27. This reduction has length at least

$$
\begin{aligned}
&2 + \mu(\rho'(A)) \\
&\quad + \mu(\rho(N_1)) \\
&\quad + \mu(\tau(P[N_1/x][\langle \mathsf{NF}(N_1), \bullet\rangle_{\rho'(A)}/\bullet])\rho(N_2) \dots \rho(N_k))
\end{aligned}
$$

and is upper bounded by $\mu(\tau(M))$. Furthermore,

$$
\begin{aligned}
&\mu(\tau((P[N_1/x])N_2 \dots N_k)) \\
&= \mu(\tau(P[N_1/x])\rho(N_2) \dots \rho(N_k)) \\
&\leq \mu(\tau(P[N_1/x][\langle \mathsf{NF}(N_1), \bullet\rangle_{\rho'(A)}/\bullet])\rho(N_2) \dots \rho(N_k))
\end{aligned}
$$

since $\bullet$ can be replaced with $\langle \mathsf{NF}(N_1), \bullet\rangle_{\rho'(A)}$ in any reduction sequence starting at

$$\tau(P[N_1/x])\pi(N_2)\ldots\pi(N_k).$$

So applying the inductive hypothesis, the above expression is lower bounded by

$$1 + \mu(A) + \mu(N_1) + \mu((P[N_1/x])N_2\ldots N_k)$$

Note that this is why we cannot simply induct over the structure of M, as we need to be able to say that

$$\mu((P[N_1/x])N_2\ldots N_k) \leq \mu(\tau((P[N_1/x])N_2\ldots N_k))$$

Finally, by Lemma 14, this implies $\mu(M) \leq \mu(\tau(M))$. □

I conclude with the main lemma of this section, which derives $(i-1)$-security from i-security.

Lemma 35. *Let $\lambda\mathcal{S}$ be an i-secure, weakly normalizing n-tiered pure type system which is i-weakly clean and not i-negatable. Then every derivable expression M such that* $\deg(M) = i-1$ *is strongly normalizing. That is, $\lambda\mathcal{S}$ is $(i-1)$-secure.*

The Negatable Case

At this point, we can prove that weak normalization implies strong normalization for all non-dependent weakly clean n-tiered pure type systems. In the previous report I have been alluding to, I use a very close variant of the translation from the previous section to given the same result in the case that s_i is negatable. In addition, for negatable sorts I give a different closure property using a novel padding technique based roughly on the notion of Skolem functions. This translation preserves typability in $\lambda\mathcal{S}$, so will not require the combinatorial-style proofs as in the previous section.

I begin with the technical lemma that makes possible the padding technique. It is based on the observation that if it is possible to abstract over a variable, then it is also possible to generalize over that variable in the types of padding variables. So a variable can commute with the padding variables *modulo generalization over that variable.* The generalization allows padding variables to instantiate their dependencies at the level of terms, which eliminates the dependence in the types.

This idea can almost eliminate the need for any notion of cleanliness, but unfortunately, the situation is complicated by products types. On the one hand, product types are simpler because they cannot be "reduced" at the level of terms except at their individual components; substitution only occurs at the type level with application steps. The difficulty is that a type of degree $i-1$ may include generalization using the rule (s_j, s_{i-1}), and it is not possible to generalize over padding variables with this rule (the types of padding variables are degree i), so we may not be able to use the Skolemization-style argument referenced above. So the translation I present still requires a cleanliness-like restriction on the rules in the system, but it is weaker.

Definition 28. *A non-dependent n-tiered pure type system $\lambda\mathcal{S}$ is i-**upwards clean** if it satisfies the following closure property: For any j (where $j \geq i$) in $[n]$, if $(s_j, s_{i-1}) \in \mathcal{R}_{\lambda\mathcal{S}}$, then $(s_j, s_i) \in \mathcal{R}_{\lambda\mathcal{S}}$ or $j \notin \mathcal{D}_{\lambda\mathcal{S}}(i)$.*

The first condition allows for the same technique for abstraction to be used for product type formation, *i.e.*, the Skolemization-style argument. The second condition is based Lemma 13, and is essentially a stronger fall-back notion of full cleanliness which captures when variables commute by virtue of the structure of the rules, as was done in the previous section. Also note that cleanliness implies i-very weak cleanliness in the case that s_i is negatable, as the existence of both the rules (s_j, s_{i-1}) and (s_i, s_i) imply the generalizability of rules (s_k, s_i) for $i \leq k \leq i-1$. This not the case for weak-cleanliness, which also shows the strength of the notion of weak harmlessness.

I continue to the above mentioned technical lemma. The following is a useful definition and notation for generalizing over many variables at a time.

Definition 29.

- *A context* Γ *is* ***generalizable over a sort*** s_j *in* $\lambda\mathcal{S}$ *if* $(s_k, s_j) \in \mathcal{R}_{\lambda\mathcal{S}}$ *for each statement* $({}^{s_k}x : A)$ *appearing in* Γ*. In abuse of notation, for a context* Γ *of the form* $(x_1 : A_1, \ldots, x_k : A_k)$*, I write* $\Pi\Gamma.\ B$ *for*

$$\Pi {x_2}^{A_1}. \ldots \Pi {x_k}^{A_k}.\ B$$

  and I write $M\Gamma$ *for* $Mx_1 \ldots x_k$*.*

- *A context* Γ *is* ***generalizable over a context*** Φ *if* Γ *is generalizable over each* s_j *such that* $({}^{s_j}x : A)$ *appears in* Φ*. In further abuse of notation, for a context* Φ *of the form* $(x_1 : B_1, \ldots, x_k : B_k)$*, I write* $\Pi\Gamma.\ \Phi$ *for*

$$(x_1 : \Pi\Gamma.\ B_1, \ldots, x_k : \Pi\Gamma.\ B_k).$$

The basic idea is the following. Suppose

$$\Gamma, x : A, y : B \vdash M(x, y) : C(x, y)$$[2]

and $\Pi x^A.\ B$ is a derivable type. It is then possible to derive

$$\Gamma, y : \Pi x^A.\ B, x : A \vdash M(x, yx) : C(x, yx).$$

2. I am using this notion to emphasize the dependence on x and y.

The variable y is replaced with yx to recover its type in the first derivation. This is a fairly simple syntactic trick, but it has some very nice consequences. The dependence on x in the type of y is represented at *the level of terms* instead of types. So if it is possible to abstract over x, then we can derive

$$\Gamma, y : \Pi x^A.\ B \vdash \lambda x^A.\ M(x, yx) : \Pi x^A.\ C(x, yx).$$

And if this λ-expression is applied to another expression N and then β-reduced, the subexpression yx is replaced with yN, maintaining the coupling of y and x without needing it to be expressed in the context.

What follows is the natural generalization of this idea. The translation below replaces variables whose types may have changed with function applications to recover their original type.

Definition 30. *For contexts Γ and Ψ, define the function $\xi^{\Gamma}_{\Psi} : \mathsf{T} \to \mathsf{T}$ as follows.*

$$\begin{aligned}
\xi^{\Gamma}_{\Psi}(s_j) &\triangleq s_j \\
\xi^{\Gamma}_{\Psi}(x) &\triangleq \begin{cases} x\Gamma & x \in \Psi \\ x & \textit{otherwise} \end{cases} \\
\xi^{\Gamma}_{\Psi}(\Pi x^A.\ B) &\triangleq \Pi x^{\xi^{\Gamma}_{\Psi}(A)}.\ \xi^{\Gamma}_{\Psi}(B) \\
\xi^{\Gamma}_{\Psi}(\lambda x^A.\ M) &\triangleq \Pi x^{\xi^{\Gamma}_{\Psi}(A)}.\ \xi^{\Gamma}_{\Psi}(M) \\
\xi^{\Gamma}_{\Psi}(MN) &\triangleq \xi^{\Gamma}_{\Psi}(M)\xi^{\Gamma}_{\Psi}(N)
\end{aligned}$$

Note that the in Ψ are irrelevant, but are included for convenience. I will write Ψ^{Γ}_{y} in the case Ψ is of the form $(y : A)$.

This translation is sufficiently simple so as to satisfy the standard substitution-commutation and β-preservation lemmas fairly straightforwardly.

Fact 6. *(ξ^{Γ}_{y} commutes with substitution) For expressions M and N, variables x and y, and context Γ,*

$$\xi^{\Gamma}_{y}(M[N/x]) = \xi^{\Gamma}_{y}(M)[\xi^{\Gamma}_{y}(N)/x].$$

Fact 7. *(ξ^{Γ}_{y} preserves β-reductions) For expressions M and N, if $M \to_\beta N$, then $\xi^{\Gamma}_{y}(M) \to_\beta \xi^{\Gamma}_{y}(N)$. Furthermore, if $M =_\beta N$, then $\xi^{\Gamma}_{y}(M) =_\beta \xi^{\Gamma}_{y}(N)$.*

And now for the lemma itself. First, the simple case in which the context on the right is a single variable. The more general case is a corollary. The proof follows the usual pattern. I also note that these results can be further generalized to the case of any pure type system, but it is simpler to present it here in the setting of tiered systems. Note that the ξ-translation is extended to contexts in the usual way.

Lemma 36. *Let Γ, Ξ, and Φ be contexts, let ${}^{s_j}x$ be variable a such that Ξ is generalizable over s_j, and let A, B and M be expressions. Then*

$$\Gamma, \Xi, {}^{s_j}x : A, \Phi \vdash M : B$$

implies

$$\Gamma, {}^{s_j}x : \Pi\Xi.\ A, \Xi, \xi^{\Xi}_{x}(\Phi) \vdash \xi^{\Xi}_{x}(M) : \xi^{\Xi}_{x}(C).$$

Proof. By induction on the structure of derivations. The cases in which the last inference is an axiom, product type formation, or an abstraction are straightforward.

Variable Introduction. First suppose $\Phi = \varnothing$. Then the last inference is of the form

$$\frac{\Gamma, \Xi \vdash A : s_j}{\Gamma, \Xi, {}^{s_j}x : A \vdash {}^{s_j}x : A}$$

Since Ξ is generalizable over s_j, it is easy to get to the derivation

$$\frac{\Gamma \vdash \Pi\Xi.\ A : s_j}{\Gamma, {}^{s_j}x : \Pi\Xi.\ A \vdash {}^{s_j}x : \Pi\Xi.\ A}$$

by a sequence of product type formations and a variable introduction. Then by thinning can derive

$$\Gamma, {}^{s_j}x : \Pi\Xi.\ A, \Xi \vdash {}^{s_j}x : \Pi\Xi.\ A$$

It is then similarly easy to derive

$$\Gamma, {}^{s_j}x : \Pi\Xi.\ A, \Xi \vdash {}^{s_j}x\Xi : A$$

by a sequence of applications. Note that x does not appear in A.

Next suppose Φ is nonempty. Then the last inference is of the form

$$\frac{\Gamma, \Xi, {}^{s_j}x : A, \Phi' \vdash B : s_k}{\Gamma, \Xi, {}^{s_j}x : A, \Phi', {}^{s_k}y : B \vdash {}^{s_k}y : B}$$

Then by induction on the length of Φ, we can derive

$$\frac{\Gamma, {}^{s_j}x : \Pi\Xi.\ A, \Xi, \xi_x^{\Xi}(\Phi') \vdash \xi_x^{\Xi}(B) : s_k}{\Gamma, {}^{s_j}x : \Pi\Xi.\ A, \Xi, \xi_x^{\Xi}(\Phi'), {}^{s_k}y : \xi_x^{\Xi}(B) \vdash {}^{s_k}y : \xi_x^{\Xi}(B)}$$

Weakening. Again, first suppose $\Phi = \varnothing$. Then the last inference is of the form

$$\frac{\Gamma, \Xi \vdash M : B \qquad \Gamma, \Xi \vdash A : s_j}{\Gamma, \Xi, {}^{s_j}x : B \vdash M : B}$$

As in the previous case, we can derive

$$\Gamma \vdash \Pi\Xi.\ A : s_j$$

which, by thinning, implies

$$\Gamma, {}^{s_j}x : \Pi\Xi.\ A, \Xi \vdash M : B$$

The case in which Φ is nonempty is similar to the analogous case for variable introduction.

Application. Suppose the last inference is of the form

$$\frac{\Gamma, \Xi, {}^{s_j}x : A, \Phi \vdash P : \Pi y^{C}.\ D \qquad \Gamma, \Xi, {}^{s_j}x : A, \Phi \vdash Q : C}{\Gamma, \Xi, {}^{s_j}x : A, \Phi \vdash PQ : D[Q/y]}$$

This case follows by applying the inductive hypothesis to each antecedant judgment and then applying substitution preservation for ξ_x^Ξ (Fact 6) to ensure that the type of the conclusion is correct.

Conversion. As with the previous case, we can apply in the inductive hypothesis to each antecedent judgment and then apply β-preservation for ξ_x^Ξ (Fact 7) to ensure that the type of the conclusion is correct. □

Corollary 3. *Let Γ, Ξ, Ψ, and Φ be contexts, such that Ξ is generalizable over Ψ (and, equivalently, over $\xi_\Psi^\Xi(\Psi)$), and let M and A be expressions. Then*

$$\Gamma, \Xi, \Psi, \Phi \vdash M : A$$

implies

$$\Gamma, \Pi\Xi.\ \xi_\Psi^\Xi(\Psi), \Xi, \xi_\Psi^\Xi(\Phi) \vdash \xi_\Psi^\Xi(M) : \xi_\Psi^\Xi(A).$$

Proof. By repeated application of Lemma 36. □

Next, we define the padding functions used in the main translation below. The types that require padding functions are those which appear as the arguments of abstractions, and these are inductively build up based on the term being translated; this accounts for the dependence on the expression M in the definition below. The dependence on the context Γ below keeps track of how the above lemma has been applied in previous steps of the translation. Also note that there are no polymorphic padding variables. Though it would be possible to include them, it is not terribly useful in combination with the Skolemization-style argument.

Definition 31. *Define the contexts $\Upsilon_{i,M}^\Gamma$ as follows by simultaneous induction on the struc-*

ture of M and length of Γ, lexicographically ordered.

$$\begin{aligned}
\Upsilon^{\Gamma}_{i,s_{i-2}} &\triangleq \varnothing \\
\Upsilon^{\Gamma}_{i,s_i x} &\triangleq \varnothing \\
\Upsilon^{\Gamma}_{i,\Pi^{s_j} x^A.\ B} &\triangleq \begin{cases} \Upsilon^{\Gamma,x:A}_{i,B} & (s_j, s_i) \in \mathcal{R}_{\lambda\mathcal{S}} \\ \Upsilon^{\Gamma}_{i,A}, \Upsilon^{\Gamma}_{i,B} & \textit{otherwise} \end{cases} \\
\Upsilon^{\Gamma}_{i,\lambda x^A.\ M} &\triangleq \mathsf{pad}^{\Gamma}_{x} : \Pi\rho'(\Gamma).\ \rho'_i(A) \to \bot_i \to \bot_i, \Upsilon^{\Gamma,x:A}_{i,M} \\
\Upsilon^{\Gamma}_{i,MN} &\triangleq \Upsilon^{\Gamma}_{i,M}, \Upsilon^{\Gamma}_{i,N}
\end{aligned}$$

Similar to what is done in the previous section, I use the shorthand

$$\langle M, N\rangle^{\Gamma}_{x} \triangleq \mathsf{pad}^{\Gamma}_{x}\rho'(\Gamma)MN$$

It is possible to characterize the dependence on Γ using the generalized Π-bindings from Definition 29.

Lemma 37. *For any expression M and contexts Γ and Φ,*

$$\Upsilon^{\Gamma,\Phi}_{M} = \Pi\Gamma.\ \Upsilon^{\Phi}_{M}.$$

Proof. By induction on the structure of M. The cases in which M is a sort, variable or application are straightforward

<u>Π-expression.</u> Suppose M is of the form $\Pi x^A.\ B$. If $(s_j, s_i) \in \mathcal{R}_{\lambda\mathcal{S}}$ then

$$\begin{aligned}
\Upsilon^{\Gamma,\Phi}_{\Pi x^A.\ B} &= \Upsilon^{\Gamma,\Phi,x:A}_{B} \\
&= \Pi\Gamma.\ \Upsilon^{\Phi,x:A}_{B} \\
&= \Pi\Gamma.\ \Upsilon^{\Phi}_{\Pi x^A.\ B}
\end{aligned}$$

The other cases are straightforward.

λ-expression. Suppose M is of the form $\lambda x^A.\ B$. Then

$$\begin{aligned}
\Upsilon^{\Gamma,\Phi}_{\lambda x^A.\ B} &= \mathsf{pad}_x : \Pi(\Gamma,\Phi).\ \rho'(A) \to \bot \to \bot, \Upsilon^{\Gamma,\Phi,x:A}_B \\
&= \mathsf{pad}_x : \Pi(\Gamma,\Phi).\ \rho'(A) \to \bot \to \bot, \Pi\Gamma.\ \Upsilon^{\Phi,x:A}_B \\
&= \Pi\Gamma.\ (\mathsf{pad}_x : \Pi\Phi.\ \rho'(A) \to \bot \to \bot, \Upsilon^{\Phi,x:A}_B) \\
&= \Pi\Gamma.\ \Upsilon^{\Phi}_{\lambda x^A.\ B}
\end{aligned}$$

□

Finally, the main translation of the section. It is similar to the translation in the previous section but with a dependence on contexts. The contexts play a similar role here as they do in the definitions above; they explicitly keep track of what needs to be ξ-translated.

Definition 32. *For each context* Γ*, define the functions*

$$\tau_i^{\Gamma} : \mathsf{T}_{i-1} \to \mathsf{T} \qquad \textit{and} \qquad \tau_i'^{\Gamma} : \mathsf{T}_{i-1} \to \mathsf{T}$$

as follows by induction on the structure of its argument and the length of Γ*, lexicographically* *ordered.*

$$
\begin{aligned}
\tau_i^{\Gamma}(s_{i-2}) &\triangleq \bullet_i \\
\tau_i^{\Gamma}({}^{s_i}x) &\triangleq {}^{s_i}x\bullet_i \\
\tau_i^{\Gamma}(\Pi x^{A}.\ B) &\triangleq \begin{cases} \tau_i^{\Gamma}(A) \to \tau_i^{\Gamma}(B) & \deg(A) = i-1 \\ \Pi x^{\rho_i'(A)}.\ \tau_i^{\Gamma,x:A}(B) & (s_j, s_i) \in \mathcal{R}_{\lambda\mathcal{S}} \\ \Pi x^{\rho_i'(A)}.\ \tau_i^{\Gamma}(B) & \textit{otherwise} \end{cases} \\
\tau_i^{\Gamma}(\lambda x^{A}.\ M) &\triangleq \lambda x^{\rho_i'(A)}.\ \tau_i'^{\Gamma,x:A}(M)\langle x, \bullet_i\rangle_x{}^{\Gamma} \\
\tau_i^{\Gamma}(MN) &\triangleq \begin{cases} \tau_i^{\Gamma}(M)\tau_i'^{\Gamma}(N) & \deg(N) = i-1 \\ \tau_i^{\Gamma}(M)\rho_i(N) & \textit{otherwise} \end{cases} \\
\tau_i'^{\Gamma}(M) &\triangleq \lambda \bullet_i{}^{\perp_i}.\ \tau_i^{\Gamma}(M)
\end{aligned}
$$

where $j \geq i-1$ *and* $\bullet_i$ *is a distinguished variable.*

As above, it is useful to characterize the dependence on Γ, this time with relation to the ξ-translation.

Lemma 38. *For contexts* Γ *and* Φ *and expression* M*,*

$$\tau^{\Gamma,\Phi}(M) = \xi^{\Gamma}_{\Upsilon^{\Phi}_{M}}(\tau^{\Phi}(M))$$

Proof. By induction on the structure M. The only case of interest is the one in which M is of the form $\lambda x^{A}.\ B$. In this case,

$$\tau^{\Gamma,\Phi}(\lambda x^{A}.\ B) = \lambda x^{\rho'(A)}.\ \tau'^{\Gamma,\Phi,x:A}(M)(\mathsf{pad}_x^{\Gamma,\Phi}\rho'(\Gamma,\Phi)x\bullet)$$

By the inductive hypothesis,

$$\tau^{\Gamma,\Phi,x:A}(M) = \xi^{\Gamma}_{\Upsilon^{\Phi,x:A}_{M}}(\tau^{\Phi}(M))$$

and since pad_x does not appear in $\tau^{\Phi}(M)$, we have

$$\xi^{\Gamma}_{\Upsilon^{\Phi,x:A}_{M}}(\tau^{\Phi}(M)) = \xi^{\Gamma}_{\Upsilon^{\Phi}_{\lambda x^A.\ M}}(\tau^{\Phi}(M))$$

It is also straightforward to check that

$$\mathsf{pad}^{\Gamma,\Phi}_{x}\rho'(\Gamma,\Phi)x\bullet = \xi^{\Gamma}_{\Upsilon^{\Phi}_{\lambda x^A.\ M}}(\mathsf{pad}^{\Phi}_{x}\rho'(\Phi)x\bullet)$$

and

$$\rho'(A) = \xi^{\Gamma}_{\Upsilon^{\Phi}_{\lambda x^A.\ M}}(\rho'(A))$$

So by definition of the ξ-translation, we have

$$\begin{aligned}
&\tau^{\Gamma,\Phi}(\lambda x^A.\ B) \\
&= \lambda x^{\rho'(A)}.\ \tau'^{\Gamma,\Phi,x:A}(M)\langle x,\bullet\rangle^{\Gamma,\Phi}_{x} \\
&= \xi^{\Gamma}_{\Upsilon^{\Phi}_{\lambda x^A.\ M}}(\lambda x^{\rho'(A)}.\ \tau'^{\Phi,x:A}(M)\langle x,\bullet\rangle^{\Phi}_{x}) \\
&= \xi^{\Gamma}_{\Upsilon^{\Phi}_{\lambda x^A.\ M}}(\tau^{\Phi}(\lambda x^A.\ B))
\end{aligned}$$

□

As in the non-negatable case, the translation must preserve typability, but this time in the same system. Note that this lemma allows for commutation with the padding variables.

Lemma 39. *Let $\lambda\mathcal{S}$ be a non-dependent tiered pure type system and suppose $\Gamma \vdash_{\lambda\mathcal{S}} M : A$ and $\Gamma \vdash_{\lambda\mathcal{S}} A : s_i$. Further suppose that Φ is a context which is generalizable over s_i. Then*

$\Gamma, \Phi \vdash_{\lambda\mathcal{S}} M : A$ *implies*

$$\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon^{\Phi}_{M}, \rho'(\Phi) \vdash_{\lambda\mathcal{S}} \tau^{\Phi}(M) : \rho(M)$$

Proof. By induction on the structure of derivations and the length of Φ, lexicographically ordered. I focus on the cases of product type formation and abstraction, which are notably different than the analogous cases in Lemma 21.

Product Type Formation. Suppose the last inference is of the form

$$\frac{\Gamma, \Phi \vdash A : s_j \qquad \Gamma, \Phi, {}^{s_j}x : A \vdash B : s_{i-1}}{\Gamma, \Phi \vdash \Pi x^A.\ B : s_{i-1}}$$

The case in which $\deg(A) = i - 1$ is similar to the analogous case in Lemma 21, so suppose $\deg(A) > i - 1$. There are two remaining cases to consider. First, if $(s_j, s_i) \in \mathcal{R}_{\lambda\mathcal{S}}$, then $\Phi, x : A$ is generalizable over s_i, which means

$$\Delta, \bullet : \bot, \rho'(\Gamma), \Upsilon^{\Phi, x:A}_{B}, \rho'(\Phi), x : \rho'(A) \vdash \tau^{\Phi, x:A}_{i}(B) : s_{i-1}$$

Otherwise, $j \notin \mathcal{D}_{\lambda\mathcal{S}}(i)$ which implies $j \notin \mathcal{D}_{\lambda\mathcal{S}}(k)$ for any rules $(s_k, s_i$ with $i \leq k \leq j$ and by an argument similar to the one for Lemma 20, the variable x does not appear free in $\Upsilon^{\varnothing}_{B}$. So by permutation and the inductive hypothesis,

$$\Delta, \bullet : \bot, \rho'(\Gamma), \rho'(\Phi), \Upsilon^{\varnothing}_{B}, x : \rho'(A) \vdash \tau(B) : s_{i-1}$$

and by product type formation, it is possible to derive

$$\Delta, \bullet : \bot, \rho'(\Gamma), \rho'(\Phi), \Upsilon^{\varnothing}_{B} \vdash \Pi x^{\rho'(A)}.\ \tau(B) : s_{i-1}$$

Finally by Corollary 3, we can derive

$$\Delta, \bullet : \bot, \rho'(\Gamma), \Pi\Phi.\ \Upsilon^{\varnothing}_{\Pi x^A.\ B}, \rho'(\Phi) \vdash \xi^{\Phi}_{\Upsilon^{\varnothing}_{\Pi x^A.\ B}}(\Pi x^{\rho'(A)}.\ \tau(B)) : s_{i-1}$$

where

$$\Pi\Phi.\ \Upsilon^{\varnothing}_{\Pi x^A.\ B} = \Upsilon^{\Phi}_{\Pi x^A.\ B}$$

by Lemma 37 and

$$\xi^{\Phi}_{\Upsilon^{\varnothing}_{\Pi x^A.\ B}}(\Pi x^{\rho'(A)}.\ \tau(B)) = \tau^{\Phi}(\Pi x^A.\ B)$$

by Lemma 38.

<u>Abstraction.</u> Suppose the last inference is of the form

$$\frac{\Gamma, \Phi, {}^{s_j}x : A \vdash M : B \qquad \Gamma, \Phi \vdash \Pi x^A.\ B : s_i}{\Gamma, \Phi \vdash \lambda x^A.\ M : \Pi x^A.\ B}$$

By the inductive hypothesis, we have

$$\Delta, \rho'(\Gamma), \rho'(\Phi) \vdash \rho'(A) : s_j$$

and

$$\Delta, \rho'(\Gamma), \Upsilon^{\Phi, x:A}_{M}, \rho'(\Phi), x : \rho'(A) \vdash \tau'^{\Phi, x:A}(M) : \bot \to \rho(A)$$

From the first we can derive by thinning and product type formation

$$\Delta, \bullet : \bot, \rho'(\Gamma) \vdash \Pi\rho'(\Phi).\ \rho'(A) \to \bot \to \bot : s_i$$

which by thinning can be added to the context of the second:

$$\begin{aligned}&\Delta, \bullet : \bot, \rho'(\Gamma), \mathsf{pad}_x : \Pi\rho'(\Phi).\ \rho'(A) \to \bot \to \bot, \Upsilon^{\Phi, x:A}_{M}, \rho'(\Phi), x : \rho'(A)\\ &\quad \vdash \tau'^{\Phi, x:A}(M) : \bot \to \rho(A)\end{aligned}$$

By thinning and repeated application, we can derive

$$\Delta, \bullet : \bot, \rho'(\Gamma), \mathsf{pad}_x : \Pi\rho'(\Phi).\ \rho'(A) \to \bot \to \bot, \Upsilon_M^{\Phi,x:A}, \rho'(\Phi), x : \rho'(A)$$
$$\vdash \mathsf{pad}_x \rho'(\Phi) x \bullet : \bot$$

and by application and abstraction we have

$$\Delta, \bullet : \bot, \rho'(\Gamma), \mathsf{pad}_x : \Pi\rho'(\Phi).\ \rho'(A) \to \bot \to \bot, \Upsilon_M^{\Phi,x:A}, \rho'(\Phi)$$
$$\vdash \lambda x^{\rho'(A)}.\ \tau'(M)\langle x, \bullet\rangle_x{}^{\Phi} : \rho(A)$$

□

The last two lemmas follow the same pattern of the previous section, but with updates in the case of product types. See also my related research report Mull [2022] for further details.

Lemma 40. *For variable ${}^{s_k}x$ and expressions M and N where $\deg(M) = i-1$ and $\deg(N) = k-1$, the following hold.*

1. *If $k = i$, then $\tau^{\Phi}(M[N/{}^{s_k}x]) \twoheadleftarrow_\beta \tau^{\Phi}(M)[\tau'^{\Phi}(N)/{}^{s_k}x]$.*
2. *If $k > i$, then $\tau^{\Phi}(M[N/{}^{s_k}x]) = \tau^{\Phi}(M)[\rho(N)/{}^{s_k}x]$.*

Lemma 41. *Let λS be a weakly normalizing non-dependent tiered pure type system. For any derivable expression M with $\deg(M) = i-1$, we have*

$$\mu(M) \leq \mu(\tau(M)) = \mu(\tau'(M))$$

And, finally, the corresponding lemma for the negatable case.

Lemma 42. *Let $\lambda\mathcal{S}$ be an i-secure weakly normalizing n-tiered pure type system which is i-negatable, and i-downward clean. Then every derivable expression M such that $\deg(M) = i-1$ is strongly normalizing. That is, $\lambda\mathcal{S}$ is $(i-1)$-secure.*

2.3.3 Main Result

I conclude this section with a statement of the main result, a special case of the Barendregt-Geuvers-Klop conjecture, which is an extention of the result of the BHS result.

Theorem 2. *Let $\lambda\mathcal{S}$ be a non-dependent n-tiered pure type system which is i-weakly clean or i-negatable and i-downward clean for each $i \in [n-1]$. If $\lambda\mathcal{S}$ is weakly normalizing then $\lambda\mathcal{S}$ is strongly normalizing.*

Proof. By reverse induction on i (from n to 0) it follows that $\lambda\mathcal{S}$ is i-secure, *i.e.*, all expressions in $\mathsf{T}_{\geq i}$ are strongly normalizing. By the classification lemma (Lemma 10), the only expressions of degree n are types and by the top-sort lemma (Lemma 7) and non-dependence, they are generated at most by s_{n-1} and Π-types using the rule (s_n, s_n) if it appears in $\mathcal{R}_{\lambda\mathcal{S}}$. It is straightforward to verify that this set of types is strongly normalizing. So suppose that $\lambda\mathcal{S}$ is k-secure. It then follows directly from Lemma 35 and Lemma 42 tha $\lambda\mathcal{S}$ is $(k-1)$-secure. □

2.4 Conclusions

I have presented an generalization of Xi's thunkification to a class of generalized non-dependent pure type systems which extends the BHS result. This class unfortunately does not contain *all* generalized non-dependent pure type sytems, but it demonstrates that progress can be made. I should note that the solving the conjecture is not so important, but rather the general questions: is it possible for a typed expression to be weakly normalizing but not strongly normalizing? What would such an expression have to look like? And, furthermore, can this normalizing behavior be reflected in another system via translation? What is are the limitations of type preserving translations of this form?

A final note on a feature which is not immediately obvious about the thunkification translation versus the CPS translation. One somewhat unfortunate feature of CPS translations is that writing down the translation itself *depends on $\lambda\mathcal{S}$ being weakly normalizing*, as the types of continuations need the type of the expression being translated. This is not the case for thunkification translations. One consequence of this fact is that the translation can be applied to non-normalizing expressions to give non-normalizing I-expressions. It is natural to wonder if this is always possible. If $\lambda\mathcal{S}$ is not weakly normalizing, is it possible to show that its I-fragment is not strongly normalizing? This may be of independent interested, as it relates to *relevance type systems*.

CHAPTER 3

A DEPENDENCY-ELIMINATING TRANSLATION FOR TIERED PURE TYPE SYSTEMS

This is work I presented in part at the 28th International Conference on Types for Proofs and Programs.

3.1 Introduction

The last few decades have seen quite a few techniques for proving normalization of systems in the λ-cube and their extensions, but many of these techniques have not been generalized to pure type systems. The purpose of the work presented in this chapter is to generalize one such technique which might be call *dependency eliminating translations*. The idea is the following: to show that a system is strongly normalizing, it suffices to define a typability-preserving infinite-reduction-sequence-preserving translation from that system into another system which is already known to be strongly normalizing. Harper et al. [1993] define such a translation from λP to $\lambda_{\rightarrow}$ and Geuvers and Nederhof [1991] extend that translation to one from λC to $\lambda\omega$. Both of these translations can be viewed as removing the *dependent* rule in the corresponding system, *i.e.*, the rule $(*, \Box)$ which allows types to depend on terms. For sufficiently well-structured pure type systems, this notion of dependence can be generalized, as is done by Barthe et al. [2001] for their definition of generalized non-dependent pure type systems (which is similar the definition of logical non-dependent pure type systems given by Coquand and Herbelin [1994]). I extend the above mentioned translations to pure type systems in a way that maintains this dependent rule elimination property. The translation can be applied to some weak sub-systems of ECC^n (defined by Luo [1990]) and can be combined with the CPS translation of Barthe et al. [2001] to provide evidence for the strong form of the Barendregt-Geuvers-Klop conjecture, which asks if the proof of strong normalization from weak normalization can be carried out in Peano arithmetic or even Heyting arithmetic.

In what follows, I present a few preliminary notions, and then I define the translation, which is given in two parts for reasons described in the corresponding section. After this is a short section on the applications of this translation.

3.2 The Translation

Recall that a tiered pure type system $\lambda\mathcal{S}$ is **non-dependent** if its rules are non-dependent, *i.e.*, if $(s, s') \in \mathcal{R}_{\lambda\mathcal{S}}$ implies $s \geq_{\mathcal{A}_{\lambda\mathcal{S}}} s'$. The translation defined in the subsequent section eliminates the dependent part of a given tiered pure type system. This motivates the following definition.

Definition 33. *The* ***non-dependent restriction*** *of a tiered pure type system λS, denoted here as λS^*, is the system specified by*

$$\mathcal{S}_{\lambda S^*} \triangleq \mathcal{S}_{\lambda S}$$
$$\mathcal{A}_{\lambda S^*} \triangleq \mathcal{A}_{\lambda S}$$
$$\mathcal{R}_{\lambda S^*} \triangleq \{(s, s', s') \in \mathcal{R}_{\lambda S} \mid (s \geq s')\}$$

For all other technical notions, I refer because to the section on definitions in the introductory chapter (Section 1.2).

I present a translation from a pure type system $\lambda\mathcal{S}$ with sufficiently rich non-dependent structure to its non-dependent restriction $\lambda\mathcal{S}^*$ which preserves typability and infinite reduction sequences. The requirements on the non-dependent structure will be quite strong, but I believe the translation to be a fairly faithful generalization of Geuvers-Nederhof translation.

At a high level, the translation eliminates dependent uses of the product type formation rule, but because it also has to preserve infinite reduction paths, it can't delete too much sub-expression information; dropping sub-expressions might eliminate redexes that account for reduction paths in pre-translated term. The rough idea is to translate all sorts down to the lowest sort s_1, so that when there is a term M of type $\Pi x^A.\ B$, the translated form of B, denoted for now as $\rho(B)$, is of sort s_1, which ensures that any product type formations with $\rho(B)$ as the target type are necessarily non-dependent (there are no sorts lower than s_1). The first issue is that translated terms must be typeable by these translated types, which indicates that there need to be two separate translations, one for types and one for terms. The second issue is that types may contain terms (*e.g.*, $(\lambda x^{s_i}.\ x)A$ for some A where $\Gamma \vdash A : s_i$) which indicates the need for separate translation for these terms as well. This process also needs to be iterated up to the top-sort, for which there are no subterms of this form by the top-sort lemma (Lemma 7).

Thus, the translation is defined in two steps. First, we define a sequence of translations that culminate in a translation ρ_0 for a types. Each translation maps sorts to a lower sort then the last, all the way down to a *type* 0 (a distinguished variable) of sort s_1. Then we define a translation γ for terms so that there is a context Γ^* where $\Gamma \vdash_{\lambda\mathcal{S}} M : A$ implies $\Gamma^* \vdash_{\lambda\mathcal{S}^*} \gamma(M) : \rho_0(A)$.

The strong requirement on the non-dependent part of λS comes from the fact that each subsequent translation includes a new variable in the context, and as more variables are included in the context, more variables need to be abstracted over so as not to lose sub-expression information. This motivates the following definition.

Definition 34. *A tiered pure type system is (i, j)-**full** if its rules contain*

$$\{(s_l, s_k) \mid l \leq i \text{ and } l \leq k \leq j\}$$

*and is **full** if it satisfies the following closure property: if $(s_i, s_j) \in \mathcal{R}_{\lambda S}$ then λS is (j, i)-full.*

We can already see how restrictive this definition is. A system which is (i, j)-full for $i \geq 2$

and $j \geq 3$ contains λU, and so is inconsistent. The strongest consistent full n-tiered system is of the has the rules

$$\{(s_k, s_1) \mid k \in [n]\} \cup \{(s_1, s_k) \mid k \in [n]\} \cup \{(s_2, s_k) \mid k \in [n]\}$$

which is a subsystem of ECC^n.

For the remainder of this section, fix a **full n-tiered pure type system $\lambda\mathcal{S}$.**

3.2.1 Type-Level Translation

The type-level translation is the last of $n+1$ inductively defined translations. Each previously defined translation is used to prove that the subsequently defined translations preserve typability. In standard abuse of notation, sequences of Πs, λs, or applications may be empty.

Definition 35. *For each i satisfying $0 \leq i \leq n$, define the function $\rho_i : \mathsf{T}_{\geq i+1} \to \mathsf{T}_{i+1}$ inductively as follows.*

$$\rho_i(s_j) \triangleq \begin{cases} 0 & i = 0 \\ s_i & \textit{otherwise} \end{cases}$$

$$\rho_i({}^{s_j}x) \triangleq {}^{s_{i+2}}x \quad (\textit{where } j \geq i+2)$$

$$\rho_i(\Pi x^A.\ B) \triangleq \Pi x^{\rho_{\deg A-1}(A)}.\ \ldots \Pi x^{\rho_i(A)}.\ \rho_i(B)$$

$$\rho_i(\lambda x^A.\ M) \triangleq \lambda x^{\rho_{\deg A-1}(A)}.\ \ldots \lambda x^{\rho_{i+1}(A)}.\ \rho_i(M)$$

$$\rho_i(MN) \triangleq \rho_i(M)\rho_{\deg N-1}(N) \ldots \rho_i(N)$$

where 0 is a distinguished variable. The definition of each translation is restricted to those cases for which there are terms in the intended domain. For example, the most basic translation $\rho_n : \mathsf{T}_{\geq n+1} \to \mathsf{T}_{n+1}$ is defined only on $\{s_n\}$ with $\rho_n(s_n) = s_n$. All other cases are *ignored. These translations are extended to contexts as follows.*

$$\rho_i(\varnothing) \triangleq (0 : s_1)$$
$$\rho_i(\Gamma, {}^{s_j}x : A) \triangleq \rho_i(\Gamma), {}^{s_j}x : \rho_{j-1}(A), \ldots, {}^{s_{i+1}}x : \rho_i(A) \qquad j \geq i+1$$

Note that $\rho_j(\Gamma) \subset \rho_i(\Gamma)$ if $i < j$, a fact which is used frequently in tandem with the thinning lemma (Lemma 4).

The following three lemmas describe the key features of the translation. First, substitution in some sense commutes the translation.

Lemma 43. *(ρ_i commutes with substitution) For all i satisfying $0 \leq i \leq n$, all j satisfying $1 \leq j \leq n$, and all terms A and B such that $A \in \mathsf{T}{\geq}\, i+1$ and $\deg B = j-1$,*

$$\rho_i(A[B/{}^{s_j}x]) = \rho_i(A)[\rho_{j-2}(B)/{}^{s_j}x] \ldots [\rho_i(B)/{}^{s_{i+2}}x]$$

Proof. By reverse induction on i. It holds trivially for ρ_n. For fixed i, we proceed by induction on the structure of A.

<u>Sort.</u> Suppose A is of the form s^k where $k \geq i$. Then for any value of j,

$$\begin{aligned}\rho_i(s_k[B/{}^{s_j}x]) &= \rho_i(s_k)\\ &= \rho_i(s_k)[\rho_{j+2}(B)/{}^{s_j}x] \ldots [\rho_i(B)/{}^{s_{i+2}}x]\end{aligned}$$

since s_k has no free variables and 0 is distinct from all other variables.

<u>Variable.</u> Suppose A is of the form ${}^{s_k}y$ where $k \geq i+2$. If $j < i+2$, then

$$\rho_i({}^{s_k}y[B/{}^{s_j}x]) = \rho_i({}^{s_k}y)$$

If $j \geq i+2$ and ${}^{s_k}y = {}^{s_j}x$ then

$$\begin{aligned}\rho_i({}^{s_j}x[B/{}^{s_j}x]) &= \rho_i(B)\\ &= {}^{s_{i+2}}x[\rho_i(B)/{}^{s_{i+2}}x]\\ &= \rho_i({}^{s_j}x)[\rho_{j-2}(B)/{}^{s_j}x]\ldots[\rho_i(B)/{}^{s_{i+2}}(x)]\end{aligned}$$

If $j \geq i+2$ and ${}^{s_k}y \neq {}^{s_j}x$ then

$$\begin{aligned}\rho_i({}^{s_k}y[B/{}^{s_j}x]) &= \rho_i({}^{s_k}y)\\ &= \rho_i({}^{s_k}y)[\rho_{j-2}(B)/{}^{s_j}x]\ldots[\rho_i(B)/{}^{s_{i+2}}(x)]\end{aligned}$$

<u>Π-Expression.</u> Suppose A is of the form $\Pi y^C.\ D$. If $\deg C < i+1$, then

$$\begin{aligned}\rho_i((\Pi y^C.\ D)[B/{}^{s_j}x]) &= \rho_i(\Pi y^{C[B/{}^{s_j}x]}.\ D[B/{}^{s_j}x])\\ &= \rho_i(D[B/{}^{s_j}x])\end{aligned}$$

If $j < i+2$ then by the inductive hypothesis

$$\rho_i(D[B/{}^{s_j}x]) = \rho_i(D) = \rho_i(\Pi x^C.\ D)$$

and likewise if $j \geq i+2$ then

$$\begin{aligned}\rho_i(D[B/{}^{s_j}x]) &= \rho_i(D)[\rho_{j-2}(B)/{}^{s_j}x]\ldots[\rho_i(B)/{}^{s_{i+2}}x]\\ &= \rho_i(\Pi x^C.\ D)[\rho_{j-2}(B)/{}^{s_j}x]\ldots[\rho_i(B)/{}^{s_{i+2}}x]\end{aligned}$$

If $\deg C \geq i+1$ then

$$\rho_i((\Pi y^C.\ D)[B/^{s_j}x])$$
$$= \rho_i(\Pi y^{C[B/^{s_j}x]}.\ D[B/^{s_j}x])$$
$$= \Pi y^{\rho_{\deg C-1}(C[B/^{s_j}x])}.\ \ldots \Pi y^{\rho_i(C[B/^{s_j}x])}.\ \rho_i(D[B/^{s_j}x])$$

If $j < i+2$ then by the inductive hypothesis

$$\rho_i(\Pi y^{C[B/^{s_j}x]}.\ D[B/^{s_j}x])$$
$$= \Pi y^{\rho_{\deg C-1}(C[B/^{s_j}x])}.\ \ldots \Pi y^{\rho_i(C[B/^{s_j}x])}.\ \rho_i(D[B/^{s_j}x])$$
$$= \Pi y^{\rho_{\deg C-1}(C)}.\ \ldots \Pi y^{\rho_i(C)}.\ \rho_i(D)$$
$$= \rho_i(\Pi y^C.\ D)$$

and likewise if $j \geq i+2$

$$\rho_i(\Pi y^{C[B/^{s_j}x]}.\ D[B/^{s_j}x])$$
$$= \Pi y^{\rho_{\deg C-1}(C[B/^{s_j}x])}.\ \ldots \Pi y^{\rho_i(C[B/^{s_j}x])}.\ \rho_i(D[B/^{s_j}x])$$
$$= \Pi y^{\rho_{\deg C-1}(C)[\rho_{j-2}(B)/^{s_j}x]\ldots[\rho_i(B)/^{s_{i+2}}x]}$$
$$\ldots \Pi y^{\rho_i(C)[\rho_{j-2}(B)/^{s_j}x]\ldots[\rho_i(B)/^{s_{i+2}}x]}$$
$$(\rho_i(D)[\rho_{j-2}(B)/^{s_j}x]\ldots[\rho_i(B)/^{s_{i+2}}x])$$
$$= (\Pi y^{\rho_{\deg C-1}(C[B/^{s_j}x])}.\ \ldots \Pi y^{\rho_i(C[B/^{s_j}x])}.\ \rho_i(D[B/^{s_j}x]))$$
$$[\rho_{j-2}(B)/^{s_j}x]\ldots[\rho_i(B)/^{s_{i+2}}x]$$
$$= \rho_i(\Pi y^C.\ D)[\rho_{j-2}(B)/^{s_j}x]\ldots[\rho_i(B)/^{s_{i+2}}x]$$

Note that here we are implicitly using the fact that if $\rho_k(C[B/^{s_j}x]) = \rho_k(C)$ then

$$\rho_k(C) = \rho_k(C)[\rho_{j-2}(B)/^{s_j}x]\ldots[\rho_i(B)/^{s_{i+2}}x]$$

<u>λ-Expression.</u>. Suppose that A is of the form $\lambda x^C.\ D$. If $\deg C < i+2$ then

$$\begin{aligned}\rho_i((\lambda x^C.\ D)[B/^{s_j}x]) &= \rho_i(\lambda x^{C[B/^{s_j}x]}.\ D[B/^{s_j}x]) \\ &= \rho_i(D[B/^{s_j}x])\end{aligned}$$

and if $\deg C \geq i+2$ then

$$\begin{aligned}&\rho_i(\lambda x^{C[B/^{s_j}x]}.\ D[B/^{s_j}x]) \\ &= \lambda x^{\rho_{\deg C-1}(C[B/^{s_j}x])}.\ \ldots \lambda x^{\rho_{i+1}(C[B/^{s_j}x])}.\ \rho_i(D[B/^{s_j}x])\end{aligned}$$

The remainder of the proof of this case is similar to the one for Π-terms.

<u>Application.</u> Suppose A is of the form MN. If $\deg N < i+1$, then

$$\begin{aligned}\rho_i((MN)[B/^{s_j}x]) &= \rho_i((M[B/^{s_j}x])(N[B/^{s_j}x])) \\ &= \rho_i(M[B/^{s_j}x])\end{aligned}$$

and if $\deg N \geq i+1$, then

$$\begin{aligned}&\rho_i((M[B/^{s_j}x])(N[B/^{s_j}x])) \\ &= \rho_i(M[B/^{s_j}x])\rho_{\deg N-1}(N[B/^{s_j}x])\ldots\rho_i(N[B/^{s_j}x])\end{aligned}$$

The remainder of the proof of this case is similar to the one for Π-terms. $\square$

The next lemma says that the translation preserves β-reductions. This is necessary for translating conversion rules; we need to be able to replace β-equivalent types after translation. Note that the lemma does not imply the translation preserves infinite reduction paths. This is because sub-expression information is lost in the translation, *e.g.*, in the case of application it may be that $\rho_i(MN) = \rho_i(M)$ and there would be no hope in preserving an infinite reduction sequence on the sub-expression N.

Lemma 44. *For all i satisfying $0 \leq i \leq n$, and all terms A and B such that $A \in \mathsf{T}{\geq}\, i+1$ and $A \twoheadrightarrow_\beta B$, it follows that $\rho_i(A) \twoheadrightarrow_\beta \rho_i(B)$.*

Proof. By reverse induction on i. This holds trivially for ρ_n. For fixed i, we proceed by induction on the definition of the multi-step β-reduction relation. In the case of the reducts:

$$\rho_i\left((\lambda x^A.\ M)N\right) \to_\beta \rho_i\left(M[N/^{s_{\deg A}}x]\right)$$

if $\deg N < i+1$ (and $\deg A < i+2$) then

$$\begin{aligned}\rho_i\left((\lambda x^A.\ M)N\right) &= \rho_i\left(\lambda x^A.\ M\right)\\ &= \rho_i(M)\\ &= \rho_i(M[^{s_{\deg A}}x/N])\end{aligned}$$

where the last equality follows from Lemma 43, and if $\deg N \geq i+1$ (and $\deg A \geq i+2$) then

$$\begin{aligned}\rho_i\left((\lambda x^A.\ M)N\right) &= \rho_i(\lambda x^A.\ M)\rho_{\deg N-1}(N)\ldots\rho_i(N)\\ &= \left(\lambda x^{\rho_{\deg A-1}(A)}.\ \ldots \lambda x^{\rho_{i+1}(A)}.\ \rho_i(M)\right)\rho_{\deg N-1}(N)\ldots\rho_i(N)\\ &\twoheadrightarrow_\beta \rho_i(M)[\rho_{\deg N-1}(N)/^{s_{\deg A}}x]\ldots[\rho_i(N)/^{s_{i+2}}x]\end{aligned}$$

Note that here we implicitly use the fact that ${}^{s_{\deg A}}x$ does not appear in A, and so

$$\rho_k(A)[\rho_l(N)/{}^{s_{l+2}}x] = \rho_k(A)$$

where $i+1 \leq k \leq \deg A - 1$ and $i \leq l \leq \deg N - 1$ so each of the subsequent β-reductions above do not change the type annotations in the associated λ-terms.

It is straightforward to verify that the translation preserves β-reductions with respect to compatibility. For example, suppose A is of the form $\Pi x^C.\ D$ and B is of the form $\Pi x^{C'}.\ D$ where $C \twoheadrightarrow_\beta C'$. If $\deg C < i+1$ then

$$\rho_i(\Pi x^C.\ D) = \rho_i(D) = \rho_i(\Pi x^{C'}.\ D)$$

If $\deg C \geq i+1$ then

$$\begin{aligned}\rho_i(\Pi x^C.\ D) &= \Pi x^{\rho_{\deg C - 1}(C)}.\ \ldots \Pi x^{\rho_i(C)}.\ \rho_i(D)\\ &\twoheadrightarrow_\beta \Pi x^{\rho_{\deg C - 1}(C')}.\ \ldots \Pi x^{\rho_i(C')}.\ \rho_i(D)\\ &= \rho_i(\Pi x^{C'}.\ D)\end{aligned}$$

where

$$\rho_i(C) \twoheadrightarrow_\beta \rho_i(C')$$

by the induction on the multi-step β-reduction relation and

$$\rho_k(C) \twoheadrightarrow_\beta \rho_k(C')$$

for $k > i$ by the induction on i. Note also that we implicitly use the fact that $\deg C = \deg C'$ (Lemma 11). □

Finally, it must be that the translation preserves typability. This ensures the translation gives well-defined types when used in combination with the term translation in the next section. Recall that $\lambda\mathcal{S}^*$ is the non-dependent restriction of $\lambda\mathcal{S}$.

Lemma 45. *For all i satisfying $0 \leq i \leq n$ the following holds. Let Γ be a context and let A and B be terms such that $A \in \mathsf{T}{\geq}\, i+1$ and $\Gamma \vdash_{\lambda\mathcal{S}} A : B$. Then*

$$\rho_{i+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \rho_i(A) : \rho_{i+1}(B)$$

Proof. By reverse induction on i. This holds trivially for ρ_n. For fixed i, it follows by induction on derivations.

<u>Axiom.</u> Suppose the derivation is of the form

$$\vdash_{\lambda\mathcal{S}} s_j : s_{j+1}$$

where $j \geq i$. If $i \geq 1$ then the translated derivation is

$$0 : s_1 \vdash_{\lambda\mathcal{S}^*} s_i : s_{i+1}$$

and otherwise

$$0 : s_1 \vdash_{\lambda\mathcal{S}^*} 0 : s_1$$

This judgment is derivable because any every full system contains the rule (s_1, s_2).

<u>Variable Introduction.</u> Suppose the last inference is of the form

$$\frac{\Gamma' \vdash_{\lambda\mathcal{S}} B : s_j}{\Gamma', {}^{s_j}x : B \vdash_{\lambda\mathcal{S}} {}^{s_j}x : B}$$

where $j \geq i+2$ (so that $\deg({}^{s_j}x) \geq i+1$). By the inductive hypothesis

$$\rho_{k+1}(\Gamma') \vdash_{\lambda\mathcal{S}^*} \rho_k(B) : s_{k+1}$$

for each k satisfying $i \leq k \leq j-1$, and since the context $\rho_{i+1}(\Gamma')$ is valid, as it appears in the case with k instantiated as i, we have

$$\rho_{i+1}(\Gamma') \vdash_{\lambda\mathcal{S}^*} \rho_k(B) : s_{k+1}$$

for each k by weakening. By further repeated weakening

$$\rho_{i+1}(\Gamma'), {}^{s_j}x : \rho_{j-1}(B), \ldots, {}^{s_{i+3}}x : \rho_{i+2}(B) \vdash_{\lambda\mathcal{S}^*} \rho_{i+1}(B) : s_{i+2}$$

and can apply the variable introduction rule:

$$\rho_{i+1}(\Gamma'), {}^{s_j}x : \rho_{j-1}(B), \ldots, {}^{s_{i+2}}x : \rho_{i+1}(B), \vdash_{\lambda\mathcal{S}^*} {}^{s_{i+2}}x : \rho_{i+1}(B)$$

Weakening. Suppose the last inference is of the form

$$\frac{\Gamma' \vdash_{\lambda\mathcal{S}} A : B \qquad \Gamma' \vdash_{\lambda\mathcal{S}} C : s_j}{\Gamma', {}^{s_j}x : C \vdash_{\lambda\mathcal{S}} A : B}$$

If $j < i+2$, then the corresponding inference is

$$\rho_{i+1}(\Gamma') \vdash_{\lambda\mathcal{S}^*} \rho_i(A) : \rho_{i+1}(B)$$

which is obtained by applying the inductive hypothesis to the left antecedent judgment. Note that the variable ${}^{s_j}x$ is dropped by the translation ρ_{i+1}. If $j \geq i+2$, we use a similar procedure as the one for the *Variable introduction* step above, *i.e.*

$$\rho_{i+1}(\Gamma') \vdash_{\lambda\mathcal{S}^*} \rho_k(C) : s_{k+1}$$

for each k satisfying $i \leq k \leq j-1$, and then by repeated weakening

$$\rho_{i+1}(\Gamma'), {}^{s_j}x : \rho_{j-1}(C), \ldots, {}^{s_{i+2}}x : \rho_{i+1}(C) \vdash_{\lambda\mathcal{S}^*} \rho_i(A) : \rho_{i+1}(B)$$

Product Type Formation. Suppose the last inference is of the form

$$\frac{\Gamma \vdash_{\lambda\mathcal{S}} C : s_j \qquad \Gamma, {}^{s_j}x : C \vdash_{\lambda\mathcal{S}} D : s_k}{\Gamma \vdash_{\lambda\mathcal{S}} \Pi x^C.\ D : s_k}$$

where $k \geq i+1$. If $j < i+1$ then

$$\rho_{i+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \rho_i(D) : s_{i+1}$$

which is obtained by applying the inductive hypothesis to the right antecedent judgment. If $j \geq i+1$, then we can derive

$$\rho_{k+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \rho_k(C) : s_{k+1}$$

for each k satisfying $i \leq k \leq j-1$, and each of these judgments can be weakened as

$$\rho_{i+1}(\Gamma), {}^{s_j}x : \rho_{j-1}(C), \ldots, {}^{s_{k+2}}x : \rho_{k+1}(C) \vdash_{\lambda\mathcal{S}^*} \rho_k(C) : s_{k+1}$$

Then with repeated uses of the product formation rule together with

$$\rho_{i+1}(\Gamma), {}^{s_j}x : \rho_{j-1}(C), \ldots, {}^{s_{i+2}}x : \rho_{i+1}(C) \vdash_{\lambda\mathcal{S}^*} \rho_i(D) : s_{i+1}$$

we can derive

$$\rho_{i+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \Pi x^{\rho_{j-1}(C)}. \ldots \Pi x^{\rho_{i+1}(C)}.\ (\rho_i(C) \to \rho_i(D)) : s_{i+1}$$

Here we use the assumption that λS^* has the rules (s_k, s_{i+1}, s_{i+1}) for each k satisfying $k \geq i+1$. Also note the use of function notation since $\rho_i(C)$ does not appear in the context above.

Abstraction.. Suppose the last inference is of the form

$$\frac{\Gamma, {}^{s_j}x : C \vdash_{\lambda\mathcal{S}} M : D \qquad \Gamma \vdash_{\lambda\mathcal{S}} \Pi x^C.\ D : s_k}{\Gamma \vdash_{\lambda\mathcal{S}} \lambda x^C.\ M : \Pi x^C.\ D}$$

where $k \geq i+2$. If $\deg C < i+2$ (*i.e.*, if $j < i+2$), then the corresponding inference is

$$\rho_{i+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \rho_i(M) : \rho_{i+1}(B)$$

since the variable ${}^{s_j}x$ is dropped by the translation ρ_{i+1}. If $\deg C \geq i+2$ then we can derive

$$\rho_{i+1}(\Gamma), {}^{s_j}x : \rho_{j-1}(C), \ldots, {}^{s_{i+2}}x : \rho_{i+1}(C) \vdash_{\lambda\mathcal{S}^*} \rho_i(M) : \rho_{i+1}(D)$$

and

$$\rho_{i+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \Pi x^{\rho_{j-1}(C)}. \ldots \Pi x^{\rho_{i+1}(C)}.\ \rho_{i+1}(D) : s_{i+2}$$

And so by repeated use of the generation lemma (Lemma 1), if $\deg C \geq i+3$, we can also derive

$$\Pi x^{\rho_k(C)}. \ldots \Pi x^{\rho_{i+1}(C)}.\ \rho_{i+1}(D) : s_{i+2}$$

in λS^* under the context

$$\rho_{i+1}(\Gamma), {}^{s_j}x : \rho_{j-1}(C), \ldots, {}^{s_{k+2}}x : \rho_{k+1}(C)$$

for each k satisfying $i+1 \leq k \leq j-2$. And so by repeated use of the abstraction rule, we can derive

$$\lambda x^{\rho_{j-1}(C)}. \ldots \lambda x^{\rho_{i+1}(C)}.\ \rho_i(M) : \Pi x^{\rho_{j-1}(C)}. \ldots \Pi x^{\rho_{i+1}(C)}.\ \rho_{i+1}(D)$$

in λS^* under the context $\rho_{i+1}(\Gamma)$.

Application. Suppose the last inference is of the form Since $\deg(MN) = \deg M$, we can apply the inductive hypothesis to the left antecedent judgment:

$$\rho_{i+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \rho_i(M) : \rho_{i+1}(\Pi x^C.\ D)$$

If $\deg N < i+1$ (and $\deg C < i+2$) then

$$\rho_{i+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \rho_i(M) : \rho_{i+1}(B)$$

If $\deg N \geq i+1$ (and $\deg C \geq i+2$) then we have

$$\rho_{i+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \rho_i(M) : \Pi x^{\rho_{\deg C-1}(C)}. \ldots \Pi x^{\rho_{i+1}(C)}.\ \rho_{i+1}(D)$$

and

$$\rho_{i+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \rho_k(N) : \rho_{k+1}(C)$$

for each k satisfying $i \leq k \leq \deg N - 1$. The first application $\rho_i(M)\rho_{\deg N-1}(N)$ has the type

$$\Pi x^{\rho_{\deg C-2}(C)[\rho_{\deg C-2}(N)/^{s_{\deg C}}x]}$$
$$\ldots \Pi x^{\rho_{i+1}(C)[\rho_{\deg C-2}(N)/^{s_{\deg C}}x]}(\rho_{i+1}(D)[\rho_{\deg N-1}(N)/^{s_{\deg C}}x])$$

The term $\rho_{\deg C-2}(C)$ does not have the variable ${}^{s_{\deg C}}x$, so this type simplifies to

$$\Pi x^{\rho_{\deg C-2}(C)}. \ldots \Pi x^{\rho_{i+1}(C)}.\ (\rho_{i+1}(D)[\rho_{\deg N-1}(N)/^{s_{\deg C}}x])$$

repeating this process, we have

$$\rho_{i+1}(\Gamma) \vdash_{\lambda S^*} \rho_i(M)\rho_{j-1}(N)\dots\rho_i(N)$$
$$: \rho_{i+1}(D)[\rho_{\deg N-1}(N)/^{s_{\deg C}}x]\dots[\rho_i(N)/^{s_{i+2}}x]$$

Since ${}^{s_{i+2}}x$ does not appear in $\rho_{i+1}(D)$, the type is

$$\rho_{i+1}(D)[\rho_{\deg N-1}(N)/^{s_{\deg C}}x]\dots[\rho_{i+1}(N)/^{s_{i+3}}x]$$

which is the same as $\rho_{i+1}(B[N/x])$ by Lemma 43.

Conversion. Suppose the last inference is of the form

$$\frac{\Gamma \vdash_{\lambda\mathcal{S}} A : C \qquad \Gamma \vdash_{\lambda\mathcal{S}} B : s_k}{\Gamma \vdash_{\lambda\mathcal{S}} A : B}$$

where $k \geq i+2$. Then the corresponding derivation is

$$\frac{\rho_{i+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \rho_i(A) : \rho_{i+1}(C) \qquad \dfrac{\rho_{i+2}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \rho_{i+1}(B) : s_{i+2}}{\rho_{i+1}(\Gamma) \vdash_{\lambda\mathcal{S}^*} \rho_{i+1}(B) : s_{i+2}}}{\rho_{i+1}\Gamma \vdash_{\lambda\mathcal{S}^*} \rho_i(A) : \rho_{i+1}(B)}$$

where $\rho_{i+1}(B) =_\beta \rho_{i+1}(C)$ by Lemma 44. □

3.2.2 Term-Level Translation

I next define the translation $\gamma : \mathsf{T} \to \mathsf{T}_0$ on all expressions. This requires one of the greater departures from the Geuvers-Nederhof translation. In their translation, variables are occasionally substituted with dummy expression via $\bot$-terms that are added to the context. This is possible since $(s_2, s_1) \in \mathcal{R}_{\lambda\omega}$. Using this technique directly would require the rules (s_{i+1}, s_i) for $i \in [n-1]$. However, by fullness, $(s_j, s_k) \in \mathcal{R}_{\lambda\mathcal{S}}$ implies $(s_l, s_1) \in \mathcal{R}_{\lambda\mathcal{S}}$ for $l \in [k]$, which will allow us a product encoding variable $\mathsf{prod} : \Pi A^{s_1}.\ 0 \to A \to 0$, which only requires the rule (s_2, s_1), which we assume appear in $\lambda\mathcal{S}$.

Definition 36. *Define the function* $\gamma : \mathsf{T} \to \mathsf{T}$ *inductively as follows.*

$$\begin{aligned}
\gamma(s_i) &\triangleq \bullet \\
\gamma({}^{s_i}x) &\triangleq {}^{s_1}x \\
\gamma(\Pi x^A.\ B) &\triangleq \mathsf{prod}(\Pi x^{\rho_{\deg A-1}(A)}.\ \ldots \Pi x^{\rho_0(A)}.\ 0)\gamma(A) \\
&\quad (\lambda x^{\rho_{\deg A-1}(A)}.\ \ldots \lambda x^{\rho_0(A)}.\ \gamma(B)) \\
\gamma(\lambda x^A.\ M) &\triangleq (\lambda z^0.\ \lambda x^{\rho_{\deg A-1}(A)}.\ \ldots \lambda x^{\rho_0(A)}.\ \gamma(M))\gamma(A) \\
\gamma(MN) &\triangleq \gamma(M)\rho_{\deg N-1}(N)\ldots\rho_0(N)\gamma(N)
\end{aligned}$$

where prod, $\bullet$ *and* z *are distinguished variables.*

We prove the same three lemmas as the previous section: typability preservation, substitution commutation, and β-preservation.

Lemma 46. *For terms* A *and* B, *if* $\Gamma \vdash_{\lambda\mathcal{S}} A : B$, *then*

$$0 : s_1, \bullet : 0, \mathsf{prod} : \Pi A^{s_1}.\ 0 \to A \to 0, \rho_0(\Gamma) \vdash_{\lambda\mathcal{S}^*} \gamma(A) : \rho_0(B)$$

Proof. By induction on derivations. The cases in which the last derivation is a variable introduction or weakening are similar to the analogous cases for Lemma 45. In what follows, let Δ denote the context $(0 : s_1, \bullet : 0, \mathsf{prod} : \Pi A^{s_1}.\ 0 \to A \to 0)$.

<u>Axioms.</u> If the derivation is of the form

$$\vdash s_i : s_{i+1}$$

then translated derivation is

$$\Delta \vdash \bullet : 0$$

In particular, the context is well-formed.

Product type formation. Suppose the last inference is of the form

$$\frac{\Gamma \vdash C : s_i \qquad \Gamma, {}^{s_i}x : C \vdash D : s_j}{\Gamma \vdash \Pi x^C .\ D : s_j}$$

By the inductive hypothesis

$$\Delta, \rho_0(\Gamma) \vdash \gamma(C) : 0$$

and

$$\Delta, \rho_0(\Gamma), {}^{s_i}x : \rho_{i-1}(C), \ldots, {}^{s_1}x : \rho_0(C) \vdash \gamma(D) : 0$$

By weakening, we can derive

$$\Delta, \rho_0(\Gamma), {}^{s_i}x : \rho_{i-1}(C), \ldots, {}^{s_1}x : \rho_0(C) \vdash 0 : s_1$$

Note that, by fullness, since $(s_i, s_j) \in \mathcal{R}_{\lambda\mathcal{S}}$, it must be that $(s_k, s_1) \in \mathcal{R}_{\lambda\mathcal{S}}$ for $k \in [i]$, and so by repeated product type formation all with the previously defined translations

$$\rho_{k+1}(\Gamma) \vdash \rho_k(C) : s_{k+1}$$

we can derive

$$\Delta, \rho_0(\Gamma), {}^{s_i}x : \rho_{i-1}(C), \ldots, {}^{s_{k+1}}x : \rho_k(C) \vdash \Pi^{s_k}x^{\rho_{k-1}(C)}. \ldots \Pi^{s_1}x^{\rho_0(C)}.\ 0 : s_1$$

where $1 \leq k \leq i$. These may be used for each abstraction step to define

$$\Delta, \rho_0(\Gamma) \vdash \lambda^{s_i}x^{\rho_{i-1}(C)}. \ldots \lambda^{s_1}x^{\rho_0(C)}.\ \gamma(D) : 0$$

Finally by two applications using the previously defined derivations, we have

$$\Delta, \rho_0(\Gamma)$$
$$\vdash \mathsf{prod}(\Pi^{s_i} x^{\rho_{i-1}(C)}. \ldots \Pi^{s_1} x^{\rho_0(C)}. 0)\gamma(A)(\lambda^{s_i} x^{\rho_{i-1}(C)}. \ldots \lambda^{s_1} x^{\rho_0(C)}. \gamma(D))0$$

Abstraction. Suppose the last inference is of the form

$$\frac{\Gamma, {}^{s_i}x : C \vdash M : D \qquad \Gamma \vdash \Pi x^C.\ D : s_j}{\Gamma \vdash \lambda x^C.\ M : \Pi x^C.\ D}$$

Similar to the same case in the proof of Lemma 45, we can derive

$$\rho_0(\Gamma) \vdash \lambda x^{\rho_{i-1}(C)}. \ldots \lambda x^{\rho_0(C)}. \gamma\, D : \Pi x^{\rho_{i-1}(C)}. \ldots \Pi x^{\rho_0(C)}. \rho_0(D)$$

and by weakening,

$$\rho_0(\Gamma), z : 0 \vdash \lambda x^{\rho_{i-1}(C)}. \ldots \lambda x^{\rho_0(C)}. \gamma\, D : \Pi x^{\rho_{i-1}(C)}. \ldots \Pi x^{\rho_0(C)}. \rho_0(D)$$

for a fresh variable z. Furthermore, we have

$$\frac{\rho_0(\Gamma) \vdash \Pi x^{\rho_{i-1}(C)}. \ldots \Pi x^{\rho_0(C)}. \rho_0(D) : s_1 \qquad \rho_0(\Gamma) \vdash 0 : s_1}{\rho_0(\Gamma) \vdash 0 \to \Pi x^{\rho_{i-1}(C)}. \ldots \Pi x^{\rho_0(C)}. \rho_0(D) : s_1}$$

so by abstraction and application,

$$\rho_0(\Gamma) \vdash (\lambda z^0.\ \lambda x^{\rho_{i-1}(C)}. \ldots \lambda x^{\rho_0(C)}. \gamma\, D)\, \gamma\, C : \Pi x^{\rho_{i-1}(C)}. \ldots \Pi x^{\rho_0(C)}. \rho_0(D)$$

Application. Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : \Pi x^C.\ D \qquad \Gamma \vdash N : C}{\Gamma \vdash MN : D[N/^{s_{\deg C}}x]}$$

If $\deg N = 0$, then we have

$$\frac{\rho_0(\Gamma) \vdash \gamma(M) : \rho_0(C) \to \rho_0(D) \qquad \rho_0(\Gamma) \vdash \gamma(N) : \tau(C)}{\rho_0(\Gamma) \vdash \gamma(M)\gamma(N) : \rho_0(D)}$$

Note that $\rho_0(D[N/^{s_1}x]) = \rho_0(D)$ by Lemma 43. The case in which $\deg N \geq 1$ is similar to the same case in the proof of Lemma 45.

Conversion. Suppose the last inference is of the form

$$\frac{\Gamma \vdash A : C \qquad \Gamma \vdash B : s_i}{\Gamma \vdash A : B}$$

It follows from Lemma 44 that $\rho_0(C) =_\beta \rho_0(B)$ so we have

$$\frac{\rho_0(\Gamma) \vdash \gamma\, A : \rho_0(C) \qquad \dfrac{\rho_1(\Gamma) \vdash \rho_0(B) : s_1}{\rho_0(\Gamma) \vdash \rho_0(B) : s_1}}{\rho_0(\Gamma) \vdash \gamma\, A : \rho_0(B)}$$

□

Next, we again show that substitution commutes with the translation. In this case, the role of this result is more auxiliary; it is necessary for the next lemma on β-reduction preservation.

Lemma 47. *For all j satisfying $1 \leq j \leq n$, and all terms A and B such that* $\deg B = j - 1$*,*

$$\gamma\, A[B/^{s_j}x] = \gamma\, A[\rho_{j-2}(B)/^{s_j}x] \ldots [\rho_0(B)/^{s_2}x][\gamma\, B/^{s_1}x]$$

Proof. By induction on the structure of A. The cases in which A is a sort or a variable are straightforward.

Π-Expressions. Suppose A is of the form $\Pi y^C.\ D$. Borrowing notation from Barendregt, in what follows, let M^+ denote $M[B/^{s_j}x]$ for any term M. Then

$$\begin{aligned}
&\gamma((\Pi y^C.\ D)^+) \\
&= \gamma(\Pi y^{C^+}.\ D^+) \\
&= \mathsf{prod}(\Pi^{s_{\deg(C^+)}} x^{\rho_{\deg(C^+)-1}(C^+)}. \ldots \Pi^{s_1} x^{\rho_0(C^+)}.\ 0)\gamma(A) \\
&\qquad (\lambda^{s_i} x^{\rho_{\deg(C^+)-1}(C^+)}. \ldots \lambda^{s_1} x^{\rho_0(C^+)}.\ \gamma(D^+))
\end{aligned}$$

Since all substitutions commute, either by the inductive hypothesis, or by Lemma 43, the desired equality follows.

The case of λ-expressions and applications are similar. $\square$

Lemma 48. *For all terms A and B, if $A \to_\beta B$, then $\gamma\, A \twoheadrightarrow_\beta^+ \gamma\, B$.*

Proof. By induction on the definition of the multi-step β-reduction relation. For the case of reducts of the form

$$(\lambda x^A.\ M)N \to_\beta M[N/^{s_{\deg A}}x]$$

If $\deg N = 0$ (and $\deg A = 1$), then

$$\begin{aligned}
\gamma\,(\lambda x^A.\ M)N &= \gamma\,\lambda x^A.\ M\,\gamma\,N \\
&= (\lambda z^0.\ \lambda x^{\rho_0(A)}.\ \gamma\,M)\,\gamma\,A\,\gamma\,N \\
&\to_\beta (\lambda x^{\rho_0(A)}.\ \gamma\,M)\,\gamma\,N \\
&\to_\beta \gamma\,M[\gamma\,N/^{s_1}x] \\
&= \gamma\,M[N/^{s_1}x]
\end{aligned}$$

If $\deg N \geq 1$, then

$$\begin{aligned}
\gamma\,(\lambda x^A.\ M)N &= \gamma\,\lambda x^A.\ M\,\rho_{\deg N-1}(N)\ldots\rho_0(N)\,\gamma\,N \\
&= (\lambda z^0.\ \lambda x^{\rho_{\deg A-1}(A)}.\ \ldots\lambda x^{\rho_0(A)}.\ \gamma\,M)\,\gamma\,A\,\rho_{\deg N-1}(N)\ldots\rho_0(N)\,\gamma\,N \\
&\to_\beta (\lambda x^{\rho_{\deg A-1}(A)}.\ \ldots\lambda x^{\rho_0(A)}.\ \gamma\,M)\,\rho_{\deg N-1}(N)\ldots\rho_0(N)\,\gamma\,N \\
&\twoheadrightarrow_\beta \gamma\,M[\rho_{\deg N-1}(N)/^{s_{\deg A}}x]\ldots[\rho_0(N)/^{s_2}x][\gamma\,N/^{s_1}x] \\
&= \gamma\,M[N/^{s_{\deg A}}x]
\end{aligned}$$

As in the proof of Lemma 44, verifying that the translation preserves β-reductions with respect to compatibility is straightforward. The key observation is that γ appears applied to every sub-expression of the pre-translated expression, so compatibility preserves *non-zero* β-reductions. □

3.2.3 Main Result

Theorem 3. *Let λS be an n-tiered pure type system that is full up to some i (where $i < n$). Then λS is strongly normalizing if and only if λS^* is strongly normalizing.*

Proof. Since all derivable terms of λS^* are also derivable in λS, if λS is strongly normalizing then so is λS^*. So suppose that λS is not strongly normalizing. Then there is an infinite reduction sequence

$$M_1 \to_\beta M_2 \to_\beta \dots$$

containing terms derivable in λS. By Lemma 46 and Lemma 48,

$$\gamma\, M_1 \twoheadrightarrow_\beta^+ \gamma\, M_2 \twoheadrightarrow_\beta^+ \dots$$

is an infinite sequence of terms derivable in λS^*, so λS^* is not strongly normalizing either. □

The only really interesting system to which this theorem can be applied are the n-tiered system with the rules

$$\{(s_k, s_1) \mid k \in [n]\} \cup \{(s_1, s_k) \mid k \in [n]\} \cup \{(s_2, s_k) \mid k \in [n]\}$$

as well as any subsystem of this one with the same non-dependent rules. Theorem 3 says that the strong normalization of this system is equivalent to the one with the rules.

$$\{(s_k, s_1) \mid k \in [n]\} \cup \{(s_2, s_2)\}$$

And though it would be possible to verify formally, a cursory inspection of the technique indicates that it is meta-theoretically weakly, and can be carried out in Peano arithmetic. So when bootstrapped with the result of Barthe et al. [2001], we get a special case of the strong Barendregt-Geuvers-Klop conjecture.

Corollary 4. *Weak normalization implies strong normalization for the n-tiered pure type system with rules*

$$\{(s_k, s_1) \mid k \in [n]\} \cup \{(s_1, s_k) \mid k \in [n]\} \cup \{(s_2, s_k) \mid k \in [n]\}$$

and, furthermore, this proof can be carried out in Peano Arithmetic.

3.3 Conclusions

In some sense, this result is a failed experiment in generalization. It is unfortunate, though somewhat interesting, that what I believe to be the most natural generalization of the Geuver-Nederhof translation breaks down very early when generalized to higher impredicative systems. All of my attempts to weaken the requirements on the non-dependent part of the underlying system were met by some issue, usually in the form of non-closure under substitution or β-equivalence. It does seem as though it should be possible to make a minor adjustment to the proof to get the same result for the n-tiered systems with rules

$$\{(s_1, s_k) \mid k \in [n]\} \cup \{(s_i, s_j) \mid 1 \leq i \leq j \leq n\}$$

as this would seem to better capture the role of the Geuver-Nederhof result.

CHAPTER 4

AN IRRELEVANCY-ELIMINATING TRANSLATION FOR TIERED PURE TYPE SYSTEMS

This work appears in the Post-Proceedings of the 28th International Conference on Types for Proofs and Programs.

4.1 Introduction

Little progress has been made on the Barendregt-Geuvers-Klop conjecture, in part because pure type systems in general are not amenable to standard techniques. Though natural, the generalization to pure type systems from the lambda cube is in some sense the most obvious one, a basic syntactic ambiguation of the inference rules which allows for maximal freedom in sort structure. The resulting systems may fail to have the meta-theoretic properties one might expect (*e.g.*, type unicity) so it is common to consider classes of systems which do maintain these properties. This was the purpose of introducing tiered pure type systems. But even in this setting, there are many systems to consider, some of which contain what amounts to "junk" structure. The primary contribution of this chapter is a translation of pure type systems which preserves typability and infinite reduction paths (I will simply write "path-preserving" from this point forward) and removes some of this irrelevant structure. By "removing structure" here, I mean that the target system of the translation is the same as the source system but with some sorts and rules removed.

Consider, for example, the system $\lambda\mathsf{HOL}$, which may be thought of as the system $\lambda\omega$ with an additional *superkind* sort $\triangle$ which allows for the introduction of kind variables that can appear in expressions but cannot be abstracted over. In $\lambda\mathsf{HOL}$, it is possible to derive

$$\mathfrak{A} : \Box \vdash_{\lambda\mathsf{HOL}} \lambda A^{\mathfrak{A}}.\ \lambda x^{A}.\ x : \Pi A^{\mathfrak{A}}.\ A \to A.$$

A judgment of this form cannot derived in $\lambda\omega$ because the variable $\mathfrak{A}$ cannot be introduced without the axiom $\vdash_{\lambda\mathsf{HOL}} \square : \triangle$. Thus, the introduction of $\triangle$ is meaningful with respect to what expressions can be derived. But both λHOL and $\lambda\omega$ are strongly normalizing. One basic observation is that there is a single expression inhabiting $\triangle$, namely $\square$. This sparsity of inhabitation can be leveraged to define a path-preserving translation from λHOL to $\lambda\omega$ and, in fact, from any pure type system with an isolated top-sort to the same system but without the top-sort. In the case of the judgment above, the variable $\mathfrak{A}$ can be instantiated at $*$ yielding the judgment

$$\vdash_{\lambda\omega} \lambda A^{*}.\ \lambda x^{A}.\ x : \Pi A^{*}.\ A \to A$$

derived which can be derived in $\lambda\omega$.

I generalize this observation in two ways. First, I define a path-preserving translation that eliminates not just top-sorts but also any sort which is *top-sort-like*. Second, I extend this translation to eliminate not just isolated sorts, but also sorts which may appear in some rules. This translation can be iteratively applied to $\lambda\mathcal{S}$ until a fixed point $\lambda\mathcal{S}^{\downarrow}$ is reached. Thus, it can be used to prove the strong normalization of systems $\lambda\mathcal{S}$ for which $\lambda\mathcal{S}^{\downarrow}$ is known to be strongly normalizing. It can also be bootstrapped with existing results for the Barendregt-Geuvers-Klop conjecture. The argument is simple: if $\lambda\mathcal{S}$ is weakly normalizing, then so is $\lambda\mathcal{S}^{\downarrow}$ since it can be embedded in $\lambda\mathcal{S}$. By assumption, $\lambda\mathcal{S}^{\downarrow}$ is strongly normalizing, and so $\lambda\mathcal{S}$ is strongly normalizing by the path-preserving translation. Bootstrapping with the result of Barthe *et al.* yields a proof of the Barendregt-Geuvers-Klop conjecture for a larger class of systems. In particular, on a technical note, $\lambda\mathcal{S}$ may have *dependent* rules and *non-negatable* sorts (see Definition 2.23 and Definition 3.1 given by Barthe et al. [2001] as well as Definition 7 for details).

This technique bears a resemblance to the one used by Roux and Doorn [2014] in their structural theory of pure type systems, which in turn resembles the techniques of Geuvers and Nederhof [1991] and Harper et al. [1993]. In all these works, a translation is defined from one pure type system into another which has fewer rules. And though it is not explicitly stated, the translation of Roux and van Doorn can be bootstrapped in the same way as described above. In fact, their translation can be used to eliminate some rules *between* tiered systems in a disjoint union whereas the translation presented here eliminates some rules *within* the individual summands in a disjoint union of tiered systems (all while preserving strong normalization).

It is important to emphasize that this result depends on the fact that the additional structure that can be handled is irrelevant and, in particular, irrelevant with respect to normalization, not derivablity or expressibility. But if we do want to prove the full conjecture, we also have to prove it for "junk" systems, ones which may not be interesting in their own right and may have rules which don't add much to the system. This result is perhaps more meaningfully interpreted in the reverse direction: the systems $\lambda\mathcal{S}^{\downarrow}$ for which the conjecture is *not* known to hold are targets for the developments of better techniques. Ideally, some technique could handle all these systems uniformly, but as of now it may be useful to further develop the theory regarding what barriers exist, and what systems beyond the lambda cube—natural or not—may be important to study.

In what follows I present the irrelevancy-eliminating translation in two parts: one part for eliminating rules and one for eliminating sorts. The final translation will be taken as the composition of these two translations. Finally, I present its application to the Barendregt-Geuvers-Klop conjecture and conclude with a short section on what it implies about the systems which remain to be studied.

4.2 Preliminaries

4.3 The Translation

Fix an n-tiered pure type system $\lambda\mathcal{S}$. I first describe the sorts which are *top-sort-like.* Recall that s is a top-sort if there is no sort s' such that $(s, s') \in \mathcal{A}$, so s_n is the only top-sort of $\lambda\mathcal{S}$. Top-sorts are interesting in part because they tend to be sparsely inhabited. A top-sort-like sort s_i which is not a top-sort has the sort s_{i+1} above it, but to ensure s_i is sparsely inhabited, s_{i+1} should not appear in any rules. We will also be interested in top-sort-like sorts which themselves do not appear in any rules.

Definition 37.

- *A sort s_i is* ***rule-isolated*** *if for all j, neither (s_j, s_i) nor (s_i, s_j) appear in $\mathcal{R}_{\lambda\mathcal{S}}$.*
- *A sort s_i is* ***top-sort-like*** *if $i < n$ implies s_{i+1} is rule-isolated (i.e., s_i is a top-sort or its succeeding sort is rule-isolated).*
- *A sort s_i is* ***completely isolated*** *if s_i is top-sort-like and rule-isolated.*

Next, I describe the structure that will be considered irrelevant with respect to normalization. Roughly speaking, this includes rules on top-sort-like sorts which allow for the derivation of redexes on expressions from sparsely inhabited types. It will be possible to essentially pre-reduce these redexes in the translation, eliminating the need for the rules in the target system of the translation. In what follows, it will be convenient to consider *sets* of top-sort-like sorts. I call a subset $\mathcal{I}$ of $[n]$ an *index set* for $\lambda\mathcal{S}$, and denote by $\mathcal{S}_\mathcal{I}$ the set $\{s_i \mid i \in \mathcal{I}\}$.

Definition 38.

- *For any index set $\mathcal{J}$, a sort s_i is* ***$\mathcal{J}$-irrelevant*** *if there is no sort s_j such that $j \in \mathcal{J}$ and $(s_j, s_i) \in \mathcal{R}_{\lambda\mathcal{S}}$. A sort s_i is* ***irrelevant*** *if it is $[n]$-irrelevant.*

- *An index set $\mathcal{I}$ is **completely irrelevant** in $\lambda\mathcal{S}$, if for each i in $\mathcal{I}$,*
  - *s_i is top-sort-like and irrelevant;*
  - *s_{i-1} is $([n] \setminus \mathcal{I})$-irrelevant.*

In the case of complete irrelevance, if $\mathcal{I}$ is a singleton set $\{i\}$, then the only rule with s_{i-1} appearing second is (s_i, s_{i-1}). By considering sets of indices simultaneously, we can make weaker assumptions on these preceding sorts. The condition of $([n] \setminus \mathcal{I})$-irrelevance ensures that s_{i-1} becomes irrelevant after removing the rules associated with sorts in $\mathcal{S}_\mathcal{I}$. Note also that if $(s_i, s_i) \in \mathcal{R}_{\lambda\mathcal{S}}$, then any completely irrelevant index set cannot contain $i-1$, i or $i+1$. Finally, it is important that there is a *unique maximum completely irrelevant index set*. In particular, the union of any two completely irrelevant index sets is completely irrelevant.

4.3.1 Eliminating Completely Irrelevant Rules

This section contains the translation which removes the rules associated with sorts whose indices appear in a completely irrelevant index set. For the remainder of the section, fix such a set $\mathcal{I}$. We begin by showing that sorts in $\mathcal{S}_\mathcal{I}$ are sparsely inhabited.

Lemma 49. *Let s_i be an irrelevant sort such that s_i is a top-sort or s_{i+1} is irrelevant. For every derivable expression A, if $\deg(A) = i$ then $A = s_{i-1}$ or $A \in \mathsf{V}_{s_{i+1}}$.*

Proof. If $i = n$, then this follows directly from the top-sort lemma (Lemma 7) and the fact that s_n is irrelevant. In fact, in this case s_n is inhabited solely by s_{n-1}. If $i \neq n$, this follows in a similar way, *i.e.*, by induction on the structure of derivations. The cases in which the last inference is an axiom, variable introduction, weakening, or conversion are straightforward. The last inference cannot be a product type formation because s_i is irrelevant. The last inference cannot be an abstraction or application because s_{i+1} is irrelevant. □

This does not hold if s_{i+1} is not irrelevant. If $(s_{i+1}, s_{i+1}) \in \mathcal{R}_{\lambda\mathcal{S}}$, for example, then $\varnothing \vdash (\lambda x^{s_i}.\ x)s_{i-1} : s_i$ is derivable. This is why we require *both* s_i and s_{i+1} to be irrelevant.

The primary challenge moving forward is dealing with the fact that variables may appear as types of sort s_i. These variables are what will necessitate s_{i+1} being not just irrelevant, but also isolated. Regardless, the sparsity of types of sort s_i induces sparsity of expressions of degree $i - 1$.

Lemma 50. *For index i in $\mathcal{I}$, context Γ and expression M, if $\Gamma \vdash M : s_{i-1}$, then M is of the form $\Pi {x_1}^{A_1}. \ldots \Pi {x_k}^{A_k}. B$ where $\deg(A_j) \in \mathcal{I}$ for all j and either $B = s_{i-2}$ or $B \in \mathsf{V}_{s_i}$.*

Proof. By induction on the structure of derivations. The cases in which the last inference is an axiom, variable introduction, or weakening are straightforward. The last inference clearly cannot be an abstraction, and it cannot be an application since s_i is irrelevant. What follows are the remaining two cases.

Product Type Formation. Suppose the last inference is of the form

$$\frac{\Gamma \vdash A : s_j \qquad \Gamma, x : A \vdash B : s_{i-1}}{\Gamma \vdash \Pi x^A. B : s_{i-1}}$$

Since s_{i-1} is $(\mathcal{S}_{\lambda\mathcal{S}} \setminus \mathcal{I})$-irrelevant, it must be that $j \in \mathcal{I}$. The desired result holds after applying the inductive hypothesis to the right antecedent judgment.

Conversion. Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash s_{i-1} : s_i}{\Gamma \vdash M : s_{i-1}}$$

where $A =_\beta s_{i-1}$. Note that $\deg(A) = i$ so $\Gamma \vdash A : s_i$ by type correctness. Thus, $A = s_{i-1}$ by Lemma 49, which means the inductive hypothesis can be applied directly to the left antecedent judgment. □

Lemma 51. *For index i in $\mathcal{I}$, context Γ, expression A and variable ${}^{s_{i+1}}x$, if $\Gamma \vdash A : {}^{s_{i+1}}x$, then $A \in \mathsf{V}_{s_i}$.*

Proof. By induction on the structure of derivations. The cases in which the last inference is an axiom, variable introduction, or weakening are straightforward. The last inference clearly cannot be a product type formation or an abstraction. The last inference cannot be an application because s_i is irrelevant. Finally, all conversions are trivial by the same argument as in the previous lemma. □

Corollary 5. *For index i in $\mathcal{I}$, every derivable expression M of degree $i-1$ is of the form $\Pi x_1{}^{A_1}. \ldots \Pi x_k{}^{A_k}. B$ where $\deg(A_j) \in \mathcal{I}$ for all j and $B = s_{i-2}$ or $B \in \mathsf{V}_{s_i}$ (and k may be 0).*

The Translation

The following translation is defined such that it essentially pre-reduces all redexes whose source types have degree in $\mathcal{I}$. Naturally, this means it does not strictly preserve β-reductions, but because these sources types are so sparsely inhabited, we can define a complexity measure on expressions which is monotonically decreasing in the β-reductions that are pre-performed by the translation. This is similar to the technique used by Sørensen [1997] for simulating π-reductions.

The other wrinkle in defining this translation is that it is difficult to pre-reduce expressions of variable type because even though such types are sparsely inhabited, it is unclear *a priori* what the value of the expression will be after a series of reductions. By Lemma 51, we know it reduces to a variable, but we don't know *which* variable, and it may be one that is generalized or abstracted over. We ensure this doesn't happen by requiring s_{i+1} is isolated, not just irrelevant. We also introduce a distinguished variable $\bullet_z$ of type z for each variable z of sort s_{i+1} in the context. This gives us a canonical term that the translation can assign to expressions of this type.

Definition 39. *Define the context-indexed function $\tau_\Gamma : \mathsf{Ctx} \times \mathsf{T} \to \mathsf{T}$ by induction on both* *arguments as follows.*

$$\tau_\Gamma(s_i) \triangleq s_i$$

$$\tau_\Gamma({}^{s_i}x) \triangleq \begin{cases} s_{i-2} & \textit{if } i \in \mathcal{I} \textit{ and } ({}^{s_i}x : s_{i-1}) \in \Gamma \\ \bullet_z & \textit{if } i \in \mathcal{I} \textit{ and } ({}^{s_i}x : {}^{s_{i+1}}z) \in \Gamma \\ {}^{s_i}x & \textit{otherwise} \end{cases}$$

$$\tau_\Gamma(\Pi x^A.\ B) \triangleq \begin{cases} \tau_{\Gamma,x:A}(B) & \deg(A) \in \mathcal{I} \\ \Pi x^{\tau_\Gamma(A)}.\ \tau_{\Gamma,x:A}(B) & \textit{otherwise} \end{cases}$$

$$\tau_\Gamma(\lambda x^A.\ M) \triangleq \begin{cases} \tau_{\Gamma,x:A}(M) & \deg(A) \in \mathcal{I} \\ \lambda x^{\tau_\Gamma(A)}.\ \tau_{\Gamma,x:A}(M) & \textit{otherwise} \end{cases}$$

$$\tau_\Gamma(MN) \triangleq \begin{cases} \tau_\Gamma(M) & \deg(N) + 1 \in \mathcal{I} \\ \tau_\Gamma(M)\tau_\Gamma(N) & \textit{otherwise} \end{cases}$$

where $\bullet_z$ is a distinguished variable. This function is used to define a function on contexts as

$$\tau(\varnothing) \triangleq \varnothing$$

$$\tau(\Gamma, {}^{s_j}x : A) \triangleq \begin{cases} \tau(\Gamma) & j \in \mathcal{I} \\ \tau(\Gamma), {}^{s_j}x : s_{j-1}, \bullet_x : {}^{s_j}x & \textit{if } j-1 \in \mathcal{I} \textit{ and } A = s_{j-1} \\ \tau(\Gamma), {}^{s_j}x : \tau_\Gamma(A) & \textit{otherwise.} \end{cases}$$

As for proving the desired features of this translation, first note if $i \in \mathcal{I}$, then the translation maps expressions of degree $i - 1$ (where $i \in \mathcal{I}$) to a sort or a $\bullet$-variable.

Proposition 5. *For any index i in $\mathcal{I}$, context Γ, and term A, if $\Gamma \vdash A : s_{i-1}$, then $\tau_\Gamma(A) =$*

s_{i-2}, *and if* $\Gamma \vdash A : {}^{s_{i+1}}x$ *for some variable* ${}^{s_{i+1}}x$*, then* $\tau_\Gamma(A) = \bullet_x$.

It suffices to consider the expressions of the form specified by Corollary 5, for which the above fact clearly holds. This turns out to be a key feature of the translation. Because the translation is able to drop so much information about these expressions, we can pre-reduce redexes in which they appear on the right.

We also use the fact that the context argument of the translation can be weakened when the last variable does not appear in the expression argument.

Proposition 6. *For any context* Γ*, expressions* M*,* A*, and* B*, and variable* x*, if* $\Gamma \vdash M : A$ *and* $\Gamma \vdash B : s_i$ *then* $\tau_{\Gamma, x:B}(M) = \tau_\Gamma(M)$.

We now prove the standard substitution-commutation and β-preservation lemmas for this translation.

Lemma 52. *For any index* i*, context* Γ*, expressions* M*,* N*,* A *and* B*, and variable* ${}^{s_i}x$*, if* $\Gamma, {}^{s_i}x : A \vdash M : B$ *and* $\Gamma \vdash N : A$ *then*

$$\tau_\Gamma(M[N/{}^{s_i}x]) = \begin{cases} \tau_{\Gamma, {}^{s_i}x:A}(M) & i \in \mathcal{I} \\ \tau_{\Gamma, {}^{s_i}x:A}(M)[\tau_\Gamma(N)/{}^{s_i}x] & \textit{otherwise.} \end{cases}$$

Proof. By induction on the structure of M. First suppose that $i \in \mathcal{I}$.
Sort. If M is of the form s_j, then $\tau_\Gamma(s_j[N/{}^{s_i}x]) = \tau_\Gamma(s_j)$.
Variable. First suppose M is of the form ${}^{s_i}x$. In particular, $A =_\beta B$, and since $\deg(A) = \deg(B) = i$, we have $A = B$ by Lemma 49. If $A = s_{i-1}$, then by Proposition 5 we have $\tau_\Gamma(N) = s_{i-2}$ and

$$\tau_\Gamma({}^{s_i}x[N/{}^{s_i}x]) = \tau_\Gamma(N) = s_{i-2} = \tau_{\Gamma, x:s_{i-1}}({}^{s_i}x).$$

Similarly, if A is of the form ${}^{s_{i+1}}y$, then $\tau_\Gamma(N) = \bullet_y$ and

$$\tau_\Gamma({}^{s_i}x[N/{}^{s_i}x]) = \tau_\Gamma(N) = \bullet_y = \tau_{\Gamma,x:y}({}^{s_i}x).$$

If M is of the form ${}^{s_j}y$ where ${}^{s_j}y \neq {}^{s_i}x$, then $\tau({}^{s_j}y[N/{}^{s_i}x]) = \tau({}^{s_j}y)$.

Π-Expression. If M is of the form $\Pi y^A.\ B$, then

$$\begin{aligned}\tau_\Gamma((\Pi y^A.\ B)[N/x]) &= \tau_\Gamma(\Pi y^{A[N/x]}.\ B[N/x]) \\ &= \begin{cases} \tau_{\Gamma,y:A}(B) & \deg(A) \in \mathcal{I} \\ \Pi y^{\tau_\Gamma(A)}.\ \tau_{\Gamma,y:A}(B) & \text{otherwise} \end{cases}\end{aligned}$$

where the last equality follows from the definition of τ and the inductive hypothesis. This also depends on Proposition 6 to show that $\tau_{\Gamma,y:A}(A) = \tau_\Gamma(A)$. The cases in which M is a λ-expression or application are similar. Furthermore, when $i \notin \mathcal{I}$, all cases are analogous. □

Before proving the β-preservation lemma, it is convenient to partition the β-reduction relation into two parts, one part which is directly preserved by the translation (β_1) and one part which is pre-reduced by the translation (β_2).

Definition 40. *Let β_2 denote the notion of reduction given by*

$$(\lambda x^A.\ M)N \to_{\beta_2} M[N/x]$$

where $\deg(A) \in \mathcal{I}$*, extended to a congruence relation in the usual way. Let β_1 denote the same notion of reduction but with* $\deg(A) \notin I$*, so that $\beta_1 \cap \beta_2 = \emptyset$ and $\beta_1 \cup \beta_2 = \beta$.*

Lemma 53. *For expressions M and N derivable in the context Γ, the following hold.*

- *If $M \to_{\beta_1} N$, then $\tau_\Gamma(M) \to_\beta \tau_\Gamma(N)$;*
- *if $M \to_{\beta_2} N$, then $\tau_\Gamma(M) = \tau_\Gamma(N)$;*

- *in particular, if $M =_\beta N$, then $\tau_\Gamma(M) =_\beta \tau_\Gamma(N)$.*

Proof. The last item follows directly from the first two. We prove the first two items by induction on the structure of the one-step β-reduction relation. In the case a redex $(\lambda x^A.\ M)N$, if $\deg(A) \notin \mathcal{I}$, then we have

$$\begin{aligned}
\tau_\Gamma((\lambda x^A.\ M)N) &= \tau_\Gamma(\lambda x^A.\ M)\tau_\Gamma(N) \\
&= (\lambda x^{\tau_\Gamma(A)}.\ \tau_{\Gamma,x:A}(M))\tau_\Gamma(N) \\
&\to_\beta \tau_{\Gamma,x:A}(M)[\tau_\Gamma(N)/x] \\
&= \tau_\Gamma(M[N/x])
\end{aligned}$$

and otherwise,

$$\begin{aligned}
\tau_\Gamma((\lambda x^A.\ M)N) &= \tau_\Gamma(\lambda x^A.\ M) \\
&= \tau_{\Gamma,x:A}(M) \\
&= \tau_\Gamma(M[N/x])
\end{aligned}$$

where the last equality in each sequence of equalities follows from the substitution-commutation lemma (Lemma 52). To show the desired result holds up to congruences, it must follow that expressions dropped by the translation are already in normal form.

<u>Π-Expression.</u> Suppose M is of the form $\Pi x^A.\ B$ and N is of the form $\Pi x^{A'}.\ B'$ where

$$\Pi x^A.\ B \to_\beta \Pi x^{A'}.\ B'$$

If $\deg(A) \notin \mathcal{I}$, then either $A \to_\beta A'$ and $B = B'$ or $B \to_\beta B'$ and $A = A'$ and the inductive hypothesis can be safely applied. If $\deg(A) \in \mathcal{I}$, then Lemma 49 implies that A is in normal form, so $A = A'$ and $B \to_\beta B'$, and the inductive hypothesis can be safely applied. The case in which M is a λ-expression is similar.

Application. Suppose M is of the form PQ and N is of the form $P'Q'$ where

$$PQ \to_{\beta_1} P'Q'$$

If $\deg(Q) + 1 \notin \mathcal{I}$, then either $P \to_\beta P'$ and $Q = Q'$ or $Q \to_\beta Q'$ and $P = P'$ and the inductive hypothesis can be safely applied. If $\deg(Q) + 1 \in \mathcal{I}$, Corollary 5 implies that Q is in normal form, so $Q = Q'$ and $P \to_\beta P'$ and the inductive hypothesis can be safely applied. □

With these two lemmas, we can now prove that the translation preserves typability. The system we translate to is defined simply as the one in which the rules associated with sorts in $\mathcal{S}_\mathcal{I}$ are dropped.

Definition 41. *The* ***irrelevance reduction*** *of an n-tiered pure type system $\lambda\mathcal{S}$, denoted here by $\lambda\mathcal{S}^-$, is the n-tiered system specified by the rules*

$$\mathcal{R}_{\lambda\mathcal{S}} \setminus \{(s_i, s_j) \mid i \in \mathcal{I} \text{ and } j \in [n]\}.$$

Lemma 54. *For context Γ and expressions M and A, if*

$$\Gamma \vdash_{\lambda\mathcal{S}} M : A \qquad \textit{then} \qquad \tau(\Gamma) \vdash_{\lambda\mathcal{S}^-} \tau_\Gamma(M) : \tau_\Gamma(A).$$

Proof. By induction on the structure of derivations.

Axiom. If the derivation is a single axiom $\vdash s_i : s_{i+1}$ then the translated derivation is the same axiom.

Variable Introduction. Suppose the last inference is of the form

$$\frac{\Gamma \vdash A : s_i}{\Gamma, {}^{s_i}x : A \vdash {}^{s_i}x : A}$$

First suppose $i \in \mathcal{I}$. If $A = s_{i-1}$, then $\tau_{\Gamma,x:s_{i-1}}(x) = s_{i-2}$ and

$$\tau(\Gamma) \vdash s_{i-2} : s_{i-1}$$

where $\tau(\Gamma)$ is well-formed by the inductive hypothesis; that is,

$$\tau(\Gamma) \vdash \tau_\Gamma(A) : s_i$$

implies $\tau(\Gamma)$ is well-formed. If A is of the form ${}^{s_{i+1}}y$, then $({}^{s_{i+1}}y : s_{i-1}) \in \Gamma$, which implies $(\bullet_y : {}^{s_{i+1}}y) \in \tau(\Gamma)$ and $\tau(\Gamma) \vdash \bullet_y : {}^{s_{i+1}}y$ where $\tau(\Gamma)$ is again well-formed by the inductive hypothesis.

Next suppose $i - 1 \in \mathcal{I}$ and $A = s_{i-1}$. By the inductive hypothesis, we can derive

$$\frac{\tau(\Gamma) \vdash s_{i-1} : s_i}{\tau(\Gamma), {}^{s_i}x : s_{i-1} \vdash {}^{s_i}x : s_{i-1}}$$

and so by weakening,

$$\frac{\tau(\Gamma), {}^{s_i}x : s_{i-1} \vdash {}^{s_i}x : s_{i-1} \qquad \tau(\Gamma), {}^{s_i}x : s_{i-1} \vdash {}^{s_i}x : s_{i-1}}{\tau(\Gamma), {}^{s_i}x : s_{i-1}, \bullet_x : {}^{s_i}x \vdash {}^{s_i}x : s_{i-1}}$$

The remaining cases are straightforward.

<u>Weakening.</u> Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s_i}{\Gamma, {}^{s_i}x : B \vdash M : A}$$

By Proposition 5, we have $\tau_{\Gamma,x:B}(M) = \tau_\Gamma(M)$. By type correctness, $\Gamma \vdash A : s_j$ for some index j, so $\tau_{\Gamma,x:B}(A) = \tau_\Gamma(A)$. So the inductive hypothesis implies

$$\tau(\Gamma) \vdash \tau_{\Gamma,x:B}(M) : \tau_{\Gamma,x:B}(A)$$

We can then use an argument similar to the one in the previous case to extend the context to $\tau(\Gamma, x : B)$.

<u>Product Type Formation.</u> Suppose the last inference is of the form

$$\frac{\Gamma \vdash A : s_i \qquad \Gamma, x : A \vdash B : s_j}{\Gamma \vdash \Pi x^A.\ B : s_j}$$

if $i \in \mathcal{I}$, then $\tau(\Gamma) = \tau(\Gamma, x : A)$ and $\tau_\Gamma(\Pi x^A.\ B) = \tau_{\Gamma,x:A}(B)$ and so $\tau(\Gamma) \vdash \tau_{\Gamma,x:A}(B) : s_j$ by the inductive hypothesis applied to the right antecedent judgment. It cannot be the case that $i - 1 \in \mathcal{I}$ and $A = s_{i-1}$ since s_i is rule-isolated in this case. The remaining case is straightforward.

Abstraction. Suppose the last inference is of the form

$$\frac{\Gamma, {}^{s_i}x : A \vdash M : B \qquad \Gamma \vdash \Pi x^A.\ B : s_j}{\Gamma \vdash \lambda x^A.\ M : \Pi x^A.\ B}$$

If $i \in \mathcal{I}$, then

$$\tau(\Gamma) = \tau(\Gamma, {}^{s_i}x : A)$$
$$\tau_\Gamma(\lambda x^A.\ M) = \tau_{\Gamma,x:A}(M)$$
$$\tau_\Gamma(\Pi x^A.\ B) = \tau_{\Gamma,x:A}(B)$$

so the desired judgment follows directly from the inductive hypothesis applied to the left antecedent judgment. Again, it cannot be the case that $i - 1 \in \mathcal{I}$ and $A = s_{i-1}$ since s_i is rule-isolated in this case. The remaining case is straightforward.

Application. Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : \Pi x^A.\ B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B[N/{}^{s_i}x]}$$

By type correctness, $\Gamma \vdash \Pi x^A.\ B : s_j$ for some sort s_j, and by generation, we have

$$\Gamma, {}^{s_i}x : A \vdash B : s_j$$

so by Lemma 52, if $i \in \mathcal{I}$ (*i.e.*, $\deg(N) + 1 \in \mathcal{I}$), then $\tau_\Gamma(MN) = \tau_\Gamma(M)$ and

$$\tau_\Gamma(B[N/{}^{s_i}x]) = \tau_{\Gamma,x:A}(B) = \tau_\Gamma(\Pi x^A.\ B).$$

The desired result then follows directly from the inductive hypothesis applied to the left antecedent judgment. And if $i \notin \mathcal{I}$, then $\tau_\Gamma(B[N/x]) = \tau_{\Gamma,x:A}(B)[\tau_\Gamma(N)/x]$ and we have

$$\frac{\tau(\Gamma) \vdash \tau_\Gamma(M) : \Pi x^{\tau_\Gamma(A)}.\ \tau_{\Gamma,x:A}(B) \qquad \tau(\Gamma) \vdash \tau_\Gamma(N) : \tau_\Gamma(A)}{\tau(\Gamma) \vdash \tau_\Gamma(M)\tau_\Gamma(N) : \tau_{\Gamma,x:A}(B)[\tau_\Gamma(N)/x]}$$

Conversion. Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s_i}{\Gamma \vdash M : B}$$

where $A =_\beta B$. Then we have

$$\frac{\tau(\Gamma) \vdash \tau_\Gamma(M) : \tau_\Gamma(A) \qquad \tau(\Gamma) \vdash \tau_\Gamma(B) : s_i}{\tau(\Gamma) \vdash \tau_\Gamma(M) : \tau_\Gamma(B)}$$

where $\tau_\Gamma(A) =_\beta \tau_\Gamma(B)$ by Lemma 53. □

It remains to show that this translation is path-preserving. The guiding observation is that β_2-reductions cannot make more "complex" redexes. We define a complexity measure which captures this observation by its being monotonically decreasing in β_2-reductions.

Definition 42. *The* ***shallow*** λ***-depth*** *of an expression M is the number of top-level λ's appearing in it, i.e., the function $\delta : \mathsf{T} \to \mathbb{N}$ is given by $\delta(\lambda x^A.\ N) \triangleq 1 + \delta(N)$ and $\delta(M) \triangleq 0$ otherwise. The* ***shallow*** λ***-depth of a redex*** *$(\lambda x^A.\ M)N$ is the shallow λ-depth of its left term $\lambda x^A.\ M$. I will simply write "depth" from this point forward.*

Definition 43. *Define $\mu : \mathsf{T} \to \mathbb{N}$ to be the function which maps an expression to the sum of the depths of its β_2-redexes, i.e.,*

$$\mu(s_i) = \mu(x) \triangleq 0$$
$$\mu(\Pi x^A.\ B) = \mu(\lambda x^A.\ B) \triangleq \mu(A) + \mu(B)$$
$$\mu(MN) \triangleq \begin{cases} \mu(M) + \mu(N) + \delta(MN) & MN \text{ is a } \beta_2\text{-redex} \\ \mu(M) + \mu(N) & \textit{otherwise.} \end{cases}$$

Finally, we prove the monotonicity lemma. It depends on the domain-full version of a result by Lévy [1978] for the untyped lambda calculus about the creation of new redexes. I give the statement of the result here without proof (See the work of Huet [1994], Xi [2006] among others for the standard definition of a residual).

Lemma 55. *For expressions M and N such that $M \to_\beta N$, if $(\lambda x^A.\ P)Q$ is a redex of N which is not a residual of a redex in M, then it is created in one of the following ways.*

1. $(\lambda y^B.\ y)(\lambda x^A.\ P)Q \to_\beta (\lambda x^A.\ P)Q$;
2. $(\lambda y^C.\ \lambda x^D.\ R)SQ \to_\beta (\lambda x^{D[S/y]}.\ R[S/y])Q$ *where* $A = D[S/y]$ *and* $P = R[S/y]$;
3. $(\lambda y^B.\ R)(\lambda x^A.\ P) \to_\beta R[\lambda x^A.\ P/y]$ *where* yQ *is a sub-expression of* R.

Lemma 56. *For derivable expressions M and N, if $M \to_{\beta_2} N$, then $\mu(M) > \mu(N)$.*

Proof. Suppose M reduces to N by reducing the β_2-redex $(\lambda x^C.\ P)Q$. By Corollary 5, the expression Q is of the form $\Pi {x_1}^{A_1}.\ \ldots \Pi {x_k}^{A_k}.\ B$ where $\deg(A_j) \in \mathcal{I}$ for all j and either $B = s_{i-2}$ or $B \in \mathsf{V}_{s_i}$. This means reducing a β_2-redex cannot duplicate existing redexes in M, so every redex has at most one residual in N. Furthermore, if N has a new β_2-redex, it is by item 2 of Lemma 55, *i.e.*, there are expressions C, D, R, and S, and variable z such that $P = \lambda z^D.\ R$ and

$$(\lambda x^C.\ \lambda z^D.\ R)QS \to_\beta (\lambda z^{D[Q/x]}.\ R[Q/x])S.$$

It is easy to verify that, because of the form of Q, only one new β-redex is created and, furthermore, $\delta(R[Q/x]) \leq \delta(R)$. This implies the new redex has smaller depth than the redex that was reduced, so even if it is a β_2-redex, the complexity of M decreases. □

The proof of the main theorem of this section is standard.

Theorem 4. *If $\lambda\mathcal{S}^-$ is strongly normalizing, then $\lambda\mathcal{S}$ is strongly normalizing.*

Proof. Suppose there is an infinite reduction sequence in $\lambda\mathcal{S}$

$$M_1 \to_\beta M_2 \to_\beta \dots$$

where M_1 is derivable in $\lambda\mathcal{S}$ from the context Γ. Since μ is monotonically decreasing in β_2-reductions (Lemma 56), there cannot be an infinite sequence of solely β_2-reductions contained in this sequence. This means there are infinitely many β_1 reductions in this sequence, which by Lemma 53 implies there infinitely many β-reductions in the reduction path

$$\tau_\Gamma(M_1) \twoheadrightarrow_\beta \tau_\Gamma(M_2) \twoheadrightarrow_\beta \dots$$

where $\tau_\Gamma(M_1)$ is derivable in $\lambda\mathcal{S}^-$ by Lemma 54. □

4.3.2 Eliminating Completely Isolated Sorts

We now handle completely isolated sorts. Recall that a sort s_i is completely isolated if s_i is top-sort-like and rule-isolated. This translation is slightly simpler than the first. It is a generalization of the observation made in the introduction that one can define a path-preserving translation from $\lambda\mathsf{HOL}$ to $\lambda\omega$, *i.e.*, one that eliminates the rule-isolated top-sort.

Fix an n-tiered pure type system $\lambda\mathcal{S}$ with $n > 2$, and a completely isolated sort s_i.[1] In essence, the following translation removes the completely isolated sort and shifts down all the sorts that might be above it. Because isolated sorts can only really be used to introduce variables into the context, the translation pre-substitutes those variables with dummy values that won't affect the normalization behavior of the expression after translation.

One notable feature of this translation is that it does not preserve the number of sorts in the system and, furthermore, does not preserve degree. It will be useful to be more careful

1. The restriction on n is a technicality that ensures the target system is nontrivial. See, for example, the variable case of Definition 44.

about variable annotations in the following definitions and lemmas.

Definition 44. *Define the context-indexed function* $\theta_\Gamma : \mathsf{Ctx} \times \mathsf{T} \to \mathsf{T}$ *inductively on both arguments as follows.*

$$\theta_\Gamma(s_j) \triangleq \begin{cases} s_j & j < i \\ s_{j-1} & \textit{otherwise} \end{cases}$$

$$\theta_\Gamma({}^{s_j}x) \triangleq \begin{cases} s_{i-2} & \textit{if } j = i \textit{ and } ({}^{s_i}x : s_{i-1}) \in \Gamma \\ {}^{s_j}x & j < i \\ {}^{s_{j-1}}x & \textit{otherwise} \end{cases}$$

$$\theta_\Gamma(\Pi^{s_j}x^A.\ B) \triangleq \Pi^{\theta_\Gamma(s_j)}x^{\theta_\Gamma(A)}.\ \theta_{\Gamma,x:A}(B)$$

$$\theta_\Gamma(\lambda^{s_j}x^A.\ M) \triangleq \lambda^{\theta_\Gamma(s_j)}x^{\theta_\Gamma(A)}.\ \theta_{\Gamma,x:A}(M)$$

$$\theta_\Gamma(MN) \triangleq \theta_\Gamma(M)\theta_\Gamma(N)$$

This function is used to define a function on contexts as

$$\theta(\varnothing) \triangleq \varnothing$$

$$\theta(\Gamma, {}^{s_j}x : A) \triangleq \begin{cases} \theta(\Gamma) & \textit{if } j = i \textit{ and } A = s_{i-1} \\ \theta(\Gamma), {}^{\theta_\Gamma(s_j)}x : \theta_\Gamma(A) & \textit{otherwise.} \end{cases}$$

As with the previous translation, contexts can be weakened without changing the value of the function (in analogy with Proposition 6 for τ_Γ). We go on to prove substitution-commutation, β-reduction preservation, and typability preservation. The proofs are similar to those in the previous sub-section and, consequently, are slightly abbreviated.

Lemma 57. *For context* Γ*, expressions* M*,* N*,* A *and* B*, and variable* ${}^{s_j}x$*, if* $j \neq i$ *and* $\Gamma, {}^{s_j}x : A \vdash M : B$ *and* $\Gamma \vdash N : A$ *then* $\theta_\Gamma(M[N/{}^{s_j}x]) = \theta_{\Gamma,{}^{s_j}x:A}(M)[\theta_\Gamma(N)/{}^{\theta_\Gamma(s_j)}x]$.

Proof. By induction on the structure of M. All cases are straightforward except the case in which M is a variable, but then the assumption that $j \neq i$ ensures the desired equality holds. $\square$

Lemma 58. *For expressions M and N derivable from Γ, if $M \to_\beta N$, then $\theta_\Gamma(M) \to_\beta \theta_\Gamma(N)$. Furthermore, if $M =_\beta N$, then $\theta_\Gamma(M) =_\beta \theta_\Gamma(N)$.*

Proof. The second part follows directly from the first, which follows by induction on the structure of the one-step β-reduction relation. In the case of a redex $(\lambda x^A.\ M)N$, we have

$$\begin{aligned}\theta_\Gamma((\lambda x^A.\ M)N) &= (\lambda x^{\theta_\Gamma(A)}.\ \theta_{\Gamma,x:A}(M))\theta_\Gamma(N) \\ &\to_\beta \theta_{\Gamma,x:A}(M)[\theta_\Gamma(N)/^{\theta_\Gamma(s_j)}x] \\ &= \theta_\Gamma(M[N/^{\theta_\Gamma(s_j)}x])\end{aligned}$$

where the last equality follows from Lemma 57, keeping in mind that $j \neq i$ since i is isolated, so the lemma can be safely applied. $\square$

Finally, typability preservation. The target system is as expected, the completely isolated sort s_i is removed and potential sorts above it are shifted down.

Definition 45. *The i-**collapse** of $\lambda\mathcal{S}$, denote here by $\lambda\mathcal{S}^*$, is the $(n-1)$-tiered systems specified by the rules $\{(\theta_\varnothing(s_j), \theta_\varnothing(s_k)) \mid (s_j, s_k) \in \mathcal{R}_{\lambda\mathcal{S}}\}$.*

Lemma 59. *For context Γ and expressions M and A where $M \neq s_{i-1}$, if*

$$\Gamma \vdash M : A \qquad \textit{then} \qquad \theta(\Gamma) \vdash \theta_\Gamma(M) : \theta_\Gamma(A).$$

Proof. By induction on the structure of derivations. The proof differs slightly depending on whether or not s_i is a top-sort. I make clear below which cases differ.

Axiom. Since $M \neq s_{i-1}$, the judgment $\varnothing \vdash \theta_\varnothing(s_j) : \theta_\varnothing(s_{j+1})$ is still an axiom.

Variable Introduction. Suppose the last inference is of the form

$$\frac{\Gamma \vdash A : s_j}{\Gamma, {}^{s_j}x : A \vdash {}^{s_j}x : A}$$

If $j = i$ and $A = s_{i-1}$, then $\theta(\Gamma) \vdash s_{i-2} : s_{i-1}$ is still derivable. Note that $\theta(\Gamma)$ can be proved to be well-formed by the inductive hypothesis. If $j < i$, then we have

$$\frac{\theta(\Gamma) \vdash \theta_\Gamma(A) : s_j}{\theta(\Gamma), {}^{s_j}x : \theta_\Gamma(A) \vdash {}^{s_j}x : \theta_\Gamma(A)}$$

If $j > i$, then in particular s_i is not a top-sort. This case is then similar to the previous one, keeping in mind that this might use the axiom (s_{i-1}, s_i) for the translated derivation *in the system* $\lambda\mathcal{S}^*$, but not in the case that s_i is a top-sort.

Weakening. This case follows directly from the fact that $\theta_{\Gamma,x:B}(M) = \theta_\Gamma(M)$ whenever M and B are derivable from Γ. It is also similar to the analogous case in the previous sub-section.

Product Type Formation. Suppose the last inference is of the form

$$\frac{\Gamma \vdash A : s_j \qquad \Gamma, {}^{s_j}x : A \vdash B : s_k}{\Gamma \vdash \Pi x^A.\ B : s_k}$$

Note that $j \neq i$ and $k \neq i$ since s_i is rule-isolated. In particular, neither A nor B are s_{i-1}. Therefore, we can apply the inductive hypothesis directly to each antecedent judgment and derive the desired consequent judgment.

Abstraction. Suppose the last inference is of the form

$$\frac{\Gamma, {}^{s_j}x : A \vdash M : B \qquad \Gamma \vdash \Pi x^A.\ B : s_k}{\Gamma \vdash \lambda x^A.\ M : \Pi x^A.\ B}$$

Note that $j \neq i$ since s_i is rule-isolated, and so $\Pi x^A.\ B$ would not be derivable. Furthermore, $B \neq s_i$ (so $M \neq s_{i-1}$) since s_i is irrelevant. Therefore, we can apply the inductive hypothesis directly to each antecedent judgment and derive the desired consequent judgment.

Application. Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : \Pi x^A.\ B \qquad \Gamma \vdash N : A}{\Gamma \vdash MN : B[N/x]}$$

Note that $\deg(A) \neq i+1$ (and in particular $N \neq s_{i-1}$), since s_{i+1} is rule-isolated. Furthermore, $\deg(A) \neq i$ (and $\deg(N) \neq i-1$) since s_i is rule-isolated. Therefore, we can apply the inductive hypothesis directly to each antecedent judgment to derive

$$\theta(\Gamma) \vdash \theta_\Gamma(M)\theta_\Gamma(N) : \theta_{\Gamma,x:A}(B)[\theta_\Gamma(N)/x]$$

where $\theta_{\Gamma,x:A}(B)[\theta_\Gamma(N)/x] = \theta_\Gamma(B[N/x])$ by Lemma 57.

Conversion. Suppose the last inference is of the form

$$\frac{\Gamma \vdash M : A \qquad \Gamma \vdash B : s_j}{\Gamma \vdash M : B}$$

If $M = s_{i-1}$, then $A =_\beta s_i =_\beta B$. Then by Lemma 49, in fact $A = B$. If $B = s_{i-1}$, then by Corollary 5 we again have $A = B$. Otherwise, by Lemma 58, $\theta_\Gamma(A) =_\beta \theta_\Gamma(B)$ and we can derive $\theta(\Gamma) \vdash \theta_\Gamma(M) : \theta_\Gamma(B)$ by the inductive hypothesis and conversion. □

Since β-reductions are simulated directly, the argument for the final theorem is straightforward.

Theorem 5. *If $\lambda\mathcal{S}^*$ is strongly normalizing then $\lambda\mathcal{S}$ is strongly normalizing.*

4.3.3 Final Translation

We can now consider the fixed-points of the above translations.

Definition 46. *Let $\tau(\lambda\mathcal{S})$ denote the fixed-point of taking the irrelevance reduction of $\lambda\mathcal{S}$ with respect to maximum completely irrelevant index sets. That is, repeat $\lambda\mathcal{S} \coloneqq \lambda\mathcal{S}^-$, taken with respect to the maximum completely irrelevant index set of $\lambda\mathcal{S}$, until its maximum completely irrelevant index set is empty.*

Definition 47. *Let $\theta(\lambda\mathcal{S})$ denote the fixed-point of taking the i-collapse of $\lambda\mathcal{S}$, where i is the maximum index of a complete isolated sort in $\lambda\mathcal{S}$, if one exists. That is, repeat $\lambda\mathcal{S} \coloneqq \lambda\mathcal{S}^*$,*

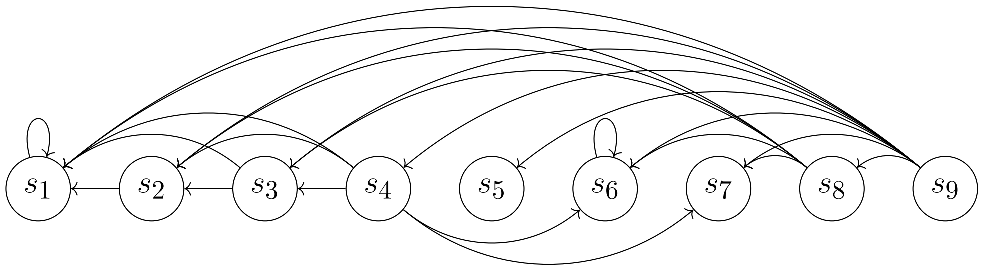


Figure 4.1: A system with a non-trivial sequence of irrelevance reductions

taken with respect to the maximum index of a completely isolated sort of $\lambda\mathcal{S}$, until it has no completely isolated sort or is 2*-tiered.*

Note that a sort which does not appear in the maximum completely irrelevant index set of $\lambda\mathcal{S}$ may appear in the maximum completely irrelevant set of $\lambda\mathcal{S}^-$. See Figure 4.1 for a tiered system with a non-trivial sequence of irrelevance reductions. The maximum completely irrelevant index set of this system is $\{9\}$, but after eliminating the rules associated with s_9, both s_9 and s_5 become rule-isolated, and so the next maximum completely irrelevant index set is $\{4, 8\}$. One can then imagine how this effect can be scaled up to larger systems.

I will write $\lambda\mathcal{S}^{\downarrow}$ for $\tau(\lambda\mathcal{S})$ and $\lambda\mathcal{S}^{\Downarrow}$ for $\theta(\tau(\lambda\mathcal{S}))$. Since no rules are remove by an i-collapse, no sort can become completely isolated and no new completely irrelevant index set can be created, so in fact $\lambda\mathcal{S}^{\Downarrow}$ is the fixed-point of $\theta \circ \tau$.

The main two theorems are as follows.

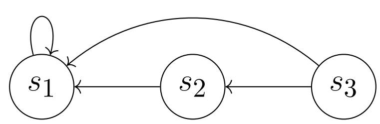
Theorem 6. *For any tiered pure type system $\lambda\mathcal{S}$, if $\lambda\mathcal{S}^{\Downarrow}$ is strongly normalizing, then $\lambda\mathcal{S}$ is strongly normalizing.*

Theorem 7. *For any tiered pure type system $\lambda\mathcal{S}$, if weak normalization implies strong normalization for $\lambda\mathcal{S}^{\downarrow}$, then weak normalization implies strong normalization for $\lambda\mathcal{S}$.*

In particular if $\lambda\mathcal{S}^{\downarrow}$ satisfies the conditions of Barthe et al. [2001] (Theorem 5.21), then weak normalization implies strong normalization in $\lambda\mathcal{S}$. This does not immediately apply to $\lambda\mathcal{S}^{\Downarrow}$ since it is not immediate that weak normalization is preserved from $\lambda\mathcal{S}^*$ to $\lambda\mathcal{S}$; the sorts are not preserved. Note that it is immediate in the case that the completely isolated sort is a top-sort. Given the scope of this work, I leave this to be verified, it is a natural step in extending these results.

4.4 Conclusions

I have presented a path-preserving translation which eliminates some irrelevant structure. Again, this structure is irrelevant with respect to normalization, not derivability. When combined with results for the Barendregt-Geuvers-Klop conjecture, it widens the class of systems for which the conjecture applies. This is a step towards proving the conjecture for all tiered systems, in particular because it highlights those systems which require further analysis. For example, it appears that dealing with circular rules is one of the clear barriers in strengthening these results. For 3-tiered systems, we extend the conjecture to (and can prove strong normalization of) the system[2]



but not to the same system with the additional rule (s_3, s_3). Circular rules break irrelevancy and, consequently, induce much more complicated structure in the system.

Additionally, it is worth noting that the conditions on completely irrelevant index sets cannot be trivially weakened. If, for example the irrelevance condition on preceding sorts was removed, this technique would apply to λU (*i.e.*, the same system presented above but with the additional rule (s_2, s_2)), leading to a contradiction since λU is non-normalizing. Circular rules again seem to be at the core of this issue. More carefully considering λU and related non-normalizing systems through the lens of these results—particularly why the

2. This system is not covered by the Barthe *et al.* result because s_2 is not negatable.

techniques don't apply to these systems—may yield a more structural understanding of the non-normalization of λU. Regardless, I hope to have demonstrated with this translation that, despite the full Barendregt-Geuvers-Klop conjecture seeming quite far from being solved, there are still a number of approachable questions and avenues for further development.

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