Asymptotic notions of computability: minimal pairs and randomness

Tiago Royer
Advisor: Denis Hirschfeldt

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Abstract

The concepts of dense computability, generic computability, coarse computability, and effective dense computability all generalize the notion of computability by requiring the algorithm to get the right answer only for "most of the inputs", rather than for all inputs (in a similar way that average-case complexity talks about expected running time, rather than imposing an upper bound in the running time of all inputs). These asymptotic notions of computability give rise to a degree structure analogous to the Turing degrees, but with different properties. In this paper we focus on minimal pairs and the level of randomness that they demand. We survey the main results in the area, and additionally settle the question of the number of minimal pairs for generic reducibility in the opposite direction that happens with the other reducibilities.

Contents

1 Introduction

The classical setting of computability and complexity theory is of worst-case scenarios; for example, the time complexity of an algorithm is the maximum number of steps, among all inputs of length n, that it takes to return an answer, and in a Turing reduction from a set A to a set B the algorithm must halt on every single input.

There is now general awareness that the strictness of worst-case scenarios may not capture a full picture of a problem or algorithm [11, 10]. Perhaps the best-known example of this phenomenon is the simplex algorithm for linear programming: there are families of instances in which the simplex algorithm takes exponential time to halt; however, on average, the algorithm converges in linear time.

1.1 Structure and Contributions of this paper

Section 2 recapitulates the basic definitions of set density, defines the four asymptotic notions of computability, and defines the corresponding notions of reducibility. Section 3 defines minimal pairs for Turing, coarse and dense degrees, and argues about the level of randomness needed to construct those degrees. Section 4 shows Igusa's result [9] that there are no minimal pairs for relative generic computability, and Hirschfeldt's result [5] that there are minimal pairs for generic reducibility.

In Section 5 we present the main result of this paper, namely, Theorem 5.4. It states that, in contrast with Theorems 3.8 and 3.14, there are very few minimal pairs for generic reducibility.

Additionally, in Section 6 we show Theorem 6.3, which is a small step towards solving Open Problem 6.1.

Finally, Section 7 catalogs the relevant open problems in the area.

1.2 Notation

We follow the convention from Computability Theory of identifying every natural number $e \in \mathbb{N}$ with the source code of a Turing machine, and vice-versa; for example, we can interpret the binary representation of $e+1$ (except the leading digit 1) as a text file containing the source code of a program in any fixed programming language. This allows us to speak of the eth Turing machine, and gives us an ordering of all Turing machines, which is useful for diagonalization arguments.

The function computed by the eth Turing machine is denoted by $\Phi_e : \mathbb{N} \to \mathbb{N}$. In the interpretation above, if the text file corresponding to e does not represent a valid program, we simply assume that Φ_e is the partial function which is nowhere defined. We assume that all machines are oracle machines, letting $\Phi_e = \Phi_e^{\emptyset}$.

Although the value of $\Phi_e(n)$ is not always defined, we can still run the eth Turing machine on n for a finite number of steps, say s; we denote this by $\Phi_e(n)[s]$, so that if $\Phi_e(n)\downarrow$ then there exists some k for which $\Phi_e(n)[s]\uparrow$ if $s < k$ and $\Phi_e(n)[s]\downarrow = \Phi_e(n)$ for all $s \geq k$.

The indicator function of a set A is denoted by $\mathbb{1}_A$ (so that $\mathbb{1}_A(x) = 1$ if $x \in A$ and $\mathbb{1}_A(x) = 0$ otherwise).

We assume that the set $\mathbb{N}$ of natural numbers contains the number 0. However, in a few cases (e.g. the sets R_k and J_k below), we will construct partitions of $\mathbb{N} \setminus \{0\}$ instead of of $\mathbb{N}$, to simplify some calculations. This means that function definitions which are based on these partitions will leave the value of $f(0)$ undefined. In all cases we can arbitrarily set $f(0) = 0$, so we omit this step of the construction.

Given two sets $A, B \in \mathbb{N}$, we define their join $A \oplus B$ by

$$A \oplus B = \{2n \mid n \in A\} \cup \{2n+1 \mid n \in B\}.$$

We identify a subset A of $\mathbb{N}$ with an infinite string in 2^ω. Therefore, we say that a finite string σ is a prefix of A, denoted $\sigma \prec A$, if the first $|\sigma|$ bits of A, interpreted as a string, agrees with σ. The basic open set $[\![\sigma]\!]$ is defined by

$$[\![\sigma]\!] = \{X \in 2^\omega \mid \sigma \prec X\}.$$

For a subset $A \subseteq \mathbb{N}$ of the natural numbers, we define $A \upharpoonright k = \{n \in A \mid n < k\}$, which is the set A "truncated" to the first k natural numbers. We can naturally identify $A \upharpoonright k$ with a binary string of length k, so in many places where $A \upharpoonright k$ is used as an argument to a computable function, the argument effectively is the corresponding string.

2 Background

This section recapitulates the basic definitions and proves some useful theorems.

2.1 Tools from Measure Theory

The following two theorems from measure theory will be used in several places in this paper.

Theorem 2.1 (Lebesgue Density Theorem). Let $\mathcal{A} \subseteq 2^\omega$ be a measurable class, and define $D(\mathcal{A})$ by

$$D(\mathcal{A}) = \{X \in 2^\omega \mid \lim_{n\to\infty} 2^n \mu([\![X \upharpoonright n]\!] \cap \mathcal{A}) = 1\}.$$

Then $D(A)$ is measurable and $\mu(\mathcal{A} \triangle D(\mathcal{A})) = 0$.

The sets $X \in D(\mathcal{A})$ are said to have *density 1* at $\mathcal{A}$.

Proof [4, Theorem 1.2.3]. Observe first that $D(\mathcal{A}) \cap D(\overline{\mathcal{A}}) = \emptyset$, so $D(\mathcal{A}) \setminus \mathcal{A} \subseteq \overline{\mathcal{A}} \setminus D(\mathcal{A})$; therefore, it suffices showing that $\mu(\mathcal{A} \setminus D(\mathcal{A})) = 0$.

For $0 < \alpha < 1$ define the set $\mathcal{B}_\alpha$ by

$$\mathcal{B}_\alpha = \{X \in \mathcal{A} \mid \lim_{n\to\infty} 2^n \mu([\![X \upharpoonright n]\!] \cap \mathcal{A}) < \alpha\}.$$

Then $\mathcal{A} \setminus D(\mathcal{A}) = \bigcup_\alpha \mathcal{B}_\alpha$, where the union is over rational $\alpha < 1$. Therefore, it suffices to show that $\mu(\mathcal{B}_\alpha) = 0$ for all $\alpha < 1$.

Suppose by contradiction that $\mu(\mathcal{B}_\alpha) > 0$. Because μ is a regular measure, there exists an open set $V \supset \mathcal{B}_\alpha$ such that $\mu(V) < \mu(\mathcal{B}_\alpha)/\alpha$. Let $U \subseteq V$ be the union of all $[\![\sigma]\!]$ for which $\mu([\![\sigma]\!] \cap \mathcal{A}) < 2^{-|\sigma|}\alpha$, and let $\sigma_0, \sigma_1, \ldots$ be a sequence of strings where the intervals $[\![\sigma_i]\!]$ are pairwise disjoint and $U = \bigcup_i [\![\sigma_i]\!]$.

On one hand, if $X \in \mathcal{B}_\alpha$, then $\mu([\![X \upharpoonright n]\!] \cap A) < \alpha 2^{-n}$ for some n, so $[\![X \upharpoonright n]\!] \subseteq U$ whence $X \in U$. This shows that $\mathcal{B}_\alpha \subseteq U$.

On the other hand,

$$\begin{aligned}
\mu(U \cap \mathcal{B}_\alpha) &= \sum_i \mu([\![\sigma_i]\!] \cap \mathcal{B}_\alpha) \\
&\leq \sum_i \mu([\![\sigma_i]\!] \cap \mathcal{A}) \\
&< \sum_i 2^{-|\sigma_i|}\alpha \\
&= \alpha\mu(U) \\
&\leq \alpha\mu(V) < \mu(\mathcal{B}_\alpha).
\end{aligned}$$

Therefore, U cannot possibly contain $\mathcal{B}_\alpha$, a contradiction. ■

A corollary of this theorem is that if $\mu(\mathcal{A}) > 0$, for any $\alpha < 1$ there exists some σ such that $\mu(\mathcal{A} \cap [\![\sigma]\!]) > \alpha 2^{-|\sigma|}$. That is, the relative density of $\mathcal{A}$ in σ can be made arbitrarily close to 1. This is the core of the "majority vote" argument, which will be used in Section 3.

A *tailset* is a class $\mathcal{A} \subseteq 2^\omega$ such that if $\sigma X \in \mathcal{A}$ then $\tau X \in \mathcal{A}$ for all τ with $|\tau| = |\sigma|$; that is, we may flip any finite number of bits of X and still stay inside $\mathcal{A}$. We have the following theorem.

Theorem 2.2 (Kolmogorov's 0-1 law). If $\mathcal{A}$ is a tailset, then either $\mu(\mathcal{A}) = 0$ or $\mu(\mathcal{A}) = 1$.

Proof. Suppose that $\mu(\mathcal{A}) > 0$, and let $\alpha < 1$ be given. Then by the above corollary to the Lebesgue Density Theorem, there exists some string σ such that $\mu(\mathcal{A} \cap [\![\sigma]\!]) > \alpha 2^{-|\sigma|}$. But because $\mathcal{A}$ is a tailset, we have $\mu(\mathcal{A} \cap [\![\sigma]\!]) = \mu(\mathcal{A} \cap [\![\tau]\!])$ for each τ with $|\tau| = |\sigma|$. This means that $\mu(\mathcal{A}) > \alpha$. Since α was arbitrary, this means $\mu(\mathcal{A}) = 1$. ■

2.2 Notions of Randomness

In this paper we will deal with three notions of randomness. The most prominent one is Martin-Löf randomness, which we define below.

Definition 2.3. Given a set B, a *Martin-Löf test* relative to B is a sequence $\{U_n\}_{n\in\mathbb{N}}$ of B-uniformly $\Sigma_1^{0,B}$ sets[1] such that $\mu(U_n) \leq 2^{-n}$ for all n. We say that a set A *passes* a Martin-Löf test $\{U_n\}_{n\in\mathbb{N}}$ if $A \notin U_n$ for some n, and we say that A is 1-*random* relative to B if A passes every Martin-Löf tests relative to B.

Sets which are 1-random relative to $\emptyset$ are simply called "1-random", or "Martin-Löf random".

Intuitively, a Martin-Löf test $\{U_n\}_{n\in\mathbb{N}}$ corresponds to a procedure for picking out regularities in sets. If $B \in U_n$, it means that U_n picked out some regularity in a prefix of B. If $B \in \bigcap_n U_n$, it means that the test can find regularities in arbitrarily long prefixes of B, so B ought not to be called random.

The definition is fairly flexible. For example, we can replace U_n with $V_n = \bigcap_{k\leq n} U_k$ to get another test with $\bigcap_n U_n = \bigcap_n V_n$, but with the additional hypothesis that $V_n \supseteq V_{n+1}$. We could require only that $\mu(U_n) \leq f(n)$ for some computable function f with $\lim_{n\to\infty} f(n) = 0$; indeed, for each n, let $V_n = \bigcap_{k\leq m} U_k$ where m is the least integer such that $f(m) \leq 2^{-n}$. Then $\{V_n\}_{n\in\mathbb{N}}$ is a Martin-Löf test and $\bigcap V_n = \bigcap U_n$.

Another possible modification is requiring the sequence $\{U_n\}_{n\in\mathbb{N}}$ to be uniformly $\Sigma_1^{0,B}$, instead of B-uniformly $\Sigma_1^{0,B}$. That is, the definition above requires that there exists a B-computable sequence $e_1, e_2, \ldots$ of indices such that

$$U_n = \{X \mid \exists k \Phi_{e_n}^B(X \upharpoonright k) = 1\}.$$

Because the Turing functional Φ_{e_n} has access to B as an oracle, we may assume that the sequence $e_1, e_2, \ldots$ is computable, rather than B-computable.

One specific class of regularities corresponds to the intuition that we should not be able to predict whether a certain bit of a sequence will be 0 or 1. We can formalize this intuition as follows.

[1] That is, there is a single B-computable sequence of indices $e_1, e_2, \ldots$ such that $U_n = \{X \mid \exists k \Phi_{e_n}^B(X \upharpoonright k) = 1\}$.

Definition 2.4. A *selection function* is a function $F : \mathbb{N} \to \mathbb{N}$ which is strictly increasing. A set A is Church-stochastic if, for every computable selection function f, we have

$$\lim_{n\to\infty} \frac{|\{k < n \mid f(k) \in A\}|}{n} = \frac{1}{2},$$

where $|C|$ denotes the cardinality of the set C.

That is, no matter how we pick the bits $f(0), f(1), \ldots$ to be analyzed, the probability of $A(f(i))$ being 1 tends to $\frac{1}{2}$. We have that 1-randomness is enough to guarantee Church-stochasticity.

Proposition 2.5. *Let A be 1-random. Then A is Church-stochastic.*

Proof. Suppose that f is a selection function for which

$$\liminf_{n\to\infty} \frac{|\{k < n \mid f(k) \in A\}|}{n} < \frac{1}{2} - \varepsilon$$

for some $\varepsilon > 0$. (If the lim sup was larger than $\frac{1}{2} + \varepsilon$, then the argument would be analogous.)

Define the set V_n by

$$V_n = \left\{ X \;\middle|\; \frac{|\{k < n \mid f(k) \in X\}|}{n} < \frac{1}{2} - \varepsilon \right\}.$$

Note that $A \in V_n$ for infinitely many n.

By the Chernoff bound [1, Theorem A.1.1],

$$\mu(V_n) \leq e^{-2\varepsilon^2 n},$$

so if we set $U_n = \bigcup_{m>n} V_n$ we have

$$\mu(U_n) \leq \frac{e^{-2\varepsilon^2 n}}{1 - e^{-2\varepsilon^2}}.$$

Since the V_n are uniformly Σ^0_1 classes, so are the U_n. By the above bound, the sequence $\{U_n\}_{n\in\mathbb{N}}$ is a Martin-Löf test. By construction, we have $A \in V_n$ for infinitely many n, which means that $A \in U_n$ for all n. This contradicts the hypothesis that A is 1-random. ∎

The converse of the above proposition is false. One striking example is provided by Ville's Theorem, which implies that there is a Church-stochastic set A such that $|\{k < n \mid k \in A\}| \leq n/2$ for all n (see e.g. [4, Theorem 6.5.1] for a proof). On the other hand, for several of our applications, Church-stochasticity will be enough.

We define the notion of n-randomness by replacing Σ^0_1 classes with Σ^0_n classes.

Definition 2.6. A set A is *n-random* relative to B if, for every B-uniformly sequence $\{U_n\}_{n\in\mathbb{N}}$ of $\Sigma^{0,B}_n$ classes such that $\mu(U_n) \leq 2^{-n}$, we have $A \notin \bigcap_n U_n$.
Sets which are n-random relative to $\emptyset$ are simply called "n-random".

Recall that the sequence $\{U_n\}_{n\in\mathbb{N}}$ is B-uniformly $\Sigma^{0,B}_n$ if there is a B-computable sequence of indices $e_1, e_2, \ldots$ such that

$$U_n = \{X \mid \exists k_1 \forall k_2 \exists k_3 \ldots Q k_n \Phi^B_{e_n}(X \restriction k_1, \ldots, X \restriction k_n) = 1\},$$

where Q is the quantifier $\exists$ if n is odd and $\forall$ if n is even. Similar to the 1-random case, we can require the sequence to be computable (as opposed to B-computable). Due to the quantifiers, we may also allow the sequence to be $B^{(n)}$-computable.

For a set A, it is true that A is $\Sigma^{0,B}_{n+1}$ if and only if A is $\Sigma^{0,B^{(n)}}_1$, but this is not the case for classes in 2^ω. Therefore, the Martin-Löf tests for n-randomness and the Martin-Löf tests for 1-randomness relative to $\emptyset^{(n-1)}$ are not the same. Surprisingly, these tests still yield the same notions of randomness. Following [4, Section 6.8], we will show this using two lemmas.

Lemma 2.7. Let n be fixed, and denote by μ_e the measure of the $\Sigma^{0,B}_n$ class indexed by e. Then the set

$$\{(e, q) \mid q \in \mathbb{Q} \wedge \mu_e > q\}$$

is a $B^{(n-1)}$-c.e. set. Thus, from an index of a $\Sigma^{0,B}_n$ class S, we can $B^{(n)}$-compute the measure $\mu(S)$.

Proof. Induction on n. For $n = 1$, given e and q, if V is the $\Sigma^{0,B}_1$ class indexed by e, then $V = \bigcup_k [\![\sigma_k]\!]$ for some sequence $\sigma_0, \sigma_1, \ldots$ of strings which can be uniformly B-computed from e. We may assume that the strings are mutually incomparable (no string is a prefix of another string), so that

$$\mu(V) = \sum_k 2^{-|\sigma_k|}.$$

Then $\mu(V) > q$ if and only if some finite portion of the sum above is greater than q, which is a B-c.e. property.

For $n > 1$, if V is the $\Sigma^{0,B}_n$ class indexed by e, we can decompose $V = \bigcup_k V_k$ where each V_k is a $\Pi^{0,B}_{n-1}$ class and $V_k \subset V_{k+1}$. Then given k and q, we can $B^{(n-1)}$-compute whether $\mu(V_k) > q$ or not. Then $\mu(V) > q$ if and only if $\mu(V_k) > q$ for some k, which is a $B^{(n-1)}$-c.e. property. ■

Lemma 2.8. The following is true for every $n \geq 1$ and every set B.[2]

1. From an index of a $\Sigma^{0,B}_n$ class S and $q \in \mathbb{Q}$, we can B-compute an index of a $\Sigma^{0,B^{(n-1)}}_1$ class U such that $U \supseteq S$ and $\mu(U) < \mu(S) + q$.

[2]This theorem is similar to [4, Theorem 6.8.3], but we simplify the proof a bit by making the claims in parts 3 and 4 a bit weaker.

2. From an index of a $\Pi^{0,B}_n$ class P and $q \in \mathbb{Q}$, we can B-compute an index of a $\Pi^{0,B^{(n-1)}}_1$ class V such that $V \subseteq S$ and $\mu(V) > \mu(P) - q$.

3. From an index of a $\Sigma^{0,B}_n$ class S and $q \in \mathbb{Q}$, we can $B^{(n)}$-compute an index of a $\Pi^{0,B}_{n-1}$ class V such that $V \subseteq S$ and $\mu(U) > \mu(S) - q$.

4. From an index of a $\Pi^{0,B}_n$ class P and $q \in \mathbb{Q}$, we can $B^{(n)}$-compute an index of a $\Sigma^{0,B}_{n-1}$ class U such that $U \supseteq S$ and $\mu(V) < \mu(P) + q$.

Proof. Note that 2 follows from 1 and that 4 follows from 3 by taking complements. We will prove 4 directly and 1 by induction.

Given a $\Sigma^{0,B}_n$ class S, let $S_0 \subseteq S_1 \subseteq \cdots$ be uniformly $\Pi^{0,B}_{n-1}$ classes such that $S = \bigcup_k S_k$.

For 3, by the lemma above, the set $B^{(n-1)}$ can compute the values of $\mu(S_k)$, so $B^{(n)}$ can compute some K for which $\mu(S_K) > \mu(S_k) - q$ for all $k > K$. Then $\mu(S_K) > \mu(S) - q$ and S_K is a $\Pi^{0,B}_{n-1}$ class contained in S.

For 1, if $n = 1$ we may let $U = S$, and if $n > 1$, use part 4 and induction to $B^{(n-1)}$-compute a sequence of indices of $\Sigma^{0,B}_{n-2}$ classes $V_0, V_1, \ldots$ such that $V_k \supseteq S_k$ and $\mu(V_k) < \mu(S_k) + q2^{-k-2}$ for all k. Now, for each k, use part 1 to compute an index for a $\Sigma^{0,B^{(n-2)}}_1$ class U_k such that $U_k \supseteq V_k$ and $\mu(U_k) < \mu(V_k) + q2^{-k-2}$. Finally, let $U = \bigcup_k U_k$. Then U is a $\Sigma^{0,B^{(n-1)}}_1$ class (because it is a union of a sequence $\Sigma^{0,B^{(n-2)}}_1$ classes, whose sequence of indices were $B^{(n-1)}$-computed from an index for S), $U \supseteq S$, and

$$\begin{aligned}
\mu(U - S) &= \mu\left(\bigcup_k U_k - \bigcup_k S_k\right) \\
&\leq \mu\left(\bigcup_k (U_k - S_k)\right) \\
&\leq \sum_k \mu(U_k - S_k) \\
&\leq \sum_k \mu(U_k - V_k) + \mu(V_k - S_k) \\
&\leq \sum_k q2^{-k-2} + q2^{-k-2} = q.
\end{aligned}$$

■

We can now prove the main result.

Theorem 2.9. A set is n-random relative to B if and only if it is 1-random relative to $B^{(n-1)}$.

Proof. Suppose that A is not n-random relative to B, and let $U_1, U_2, \ldots$ be a test for which A fails. That is, $U_1, U_2, \ldots$ is a B-uniform sequence of $\Sigma^{0,B}_n$ classes for which $\mu(U_k) < 2^{-k}$, and $A \in \bigcap_k U_k$.

By Lemma 2.8, we can B-compute indices for $\Sigma_1^{0,B^{(n-1)}}$ classes $V_1, V_2, \ldots$ with $V_k \supset U_k$ and $\mu(V_k) < \mu(U_k) + 2^{-k}$ for all k. Then $\mu(V_{k+1}) < 2^{-k}$, and $A \in \bigcap_k V_{k+1}$, which shows that A is not 1-random relative to $B^{(n-1)}$.

Conversely, suppose that A is not 1-random relative to $B^{(n-1)}$, and let $U_1, U_2, \ldots$ be a $B^{(n-1)}$-uniformly $\Sigma_1^{0,B^{(n-1)}}$ classes with $\mu(U_k) < 2^{-k}$ and $A \notin \bigcap_k U_k$. Each U_k is also a $\Sigma_n^{0,B}$ class, and B can uniformly transform an index of U_k into an index of a $\Sigma_n^{0,B}$ class V_k with $V_k = U_k$. Then because $A \notin \bigcap V_k$, we have that A is not n-random relative to B. ∎

There is a third notion of randomness which we will use in the paper, which arises by replacing the decreasing sequence of $\Sigma_n^{0,B}$ classes with a single $\Pi_n^{0,B}$ class of measure zero.

Definition 2.10. A set A is *weakly n-random* (or Kurtz n-random) relative to B if A is contained in every $\Sigma_n^{0,B}$ class of measure 1, or, equivalently, if A is not contained in any $\Pi_n^{0,B}$ class of measure 0.

If $U_1, U_2, \ldots$ is a sequence of $\Sigma_n^{0,B}$ classes, then $\bigcap_k U_k$ is a $\Pi_{n+1}^{0,B}$ class, so every weakly $(n+1)$-random set is n-random. Similarly, if V is a $\Pi_n^{0,B}$ class with measure zero, by Lemma 2.8 item 4 there exists a sequence of $B^{(n)}$-uniformly $\Sigma_{n-1}^{0,B}$ classes $U_1, U_2, \ldots$ (and thus a sequence of $B^{(n)}$-uniformly $\Sigma_n^{0,B}$ classes) with $\mu(U_1) < 2^{-n}$, so every n-random set is also weakly n-random.

We note that being weakly n-random is not the same as being weakly 1-random relative to $\emptyset^{(n-1)}$; see [4, p. 286] for a proof. But we can get a similar result if we allow for an extra quantifier.

Theorem 2.11. Let $n \geq 2$. Then A is weakly n-random relative to B if and only if A is weakly 2-random relative to $B^{(n-2)}$.

Proof. If A is not weakly 2-random relative to $B^{(n-2)}$, then A is contained in some $\Sigma_2^{0,B^{(n-2)}}$ class U of measure 1. Because U is also a $\Sigma_n^{0,B}$ class, this means that A is not weakly n-random relative to B.

Conversely, let $A \notin U$ where U is a $\Sigma_n^{0,B}$ class of measure 1. Write $U = \bigcup_k U_k$, where the U_k are uniformly $\Pi_{n-1}^{0,B}$ classes, and use Theorem 2.8 item 2 to B-uniformly get $\Pi_1^{0,B^{(n-2)}}$ classes $V_{k,j}$ such that $V_{k,j} \subseteq U_k$ and $\mu(V_{k,j}) > \mu(U_k) - 2^{-j}$. Then $V = \bigcup_{k,j}$ is a $\Sigma_2^{0,B^{(n-2)}}$ class of measure 1, and $A \notin V$, so A is not 2-random relative to $B^{(n-2)}$. ∎

2.3 Density of Sets

Definition 2.12. For $A \subseteq \mathbb{N}$, we define

$$\rho_n(A) = \frac{|A \upharpoonright n|}{n}.$$

The *upper density* and *lower density* of A, denoted by $\overline{\rho}(A)$ and $\underline{\rho}(A)$, respectively, are the limits

$$\overline{\rho}(A) = \limsup_{n\to\infty} \rho_n(A) \qquad \text{and} \qquad \underline{\rho}(A) = \liminf_{n\to\infty} \rho_n(A)$$

If these two limits coincide, this common number is the *density* of A, denoted by $\rho(A)$. We say that A is *dense* if $\rho(A) = 1$ and *sparse* if $\rho(A) = 0$.

For example, $\rho(\mathbb{N}) = 1$, the density of the set of even numbers is $\frac{1}{2}$, the density of the set of primes is 0, and, for the set

$$\{n \mid \text{there exists a Hadamard matrix of order } n\}$$

it is still an open problem whether it has positive density or not [3].

Of course, $\rho(A)$ does not necessarily exist. If α, β are any two real numbers with $0 \leq \alpha \leq \beta \leq 1$, we can construct a set A with $\underline{\rho}(A) = \alpha$ and $\overline{\rho}(A) = \beta$ by stages. Start defining $A(0) = 0$, so that at the beginning of stage s we already defined $A \upharpoonright s$. Then alternate between defining $A(s) = 1$ until $\rho_s(A) \geq \beta$, and $\rho_s(A) = 0$ until $\rho_s(A) \leq \alpha$.

This notion of density is natural, and most of our theorems will be stated using this definition. However, for proofs, it will be sometimes more convenient to use the following variant.

Definition 2.13. Let $J_k = [2^k, 2^{k+1}) \cap \mathbb{N}$.[3] If $A \subseteq \mathbb{N}$, we define

$$d_k(A) = \frac{|A \cap J_k|}{2^k},$$

and, analogously to the definition of ρ, we define

$$\underline{d}(A) = \liminf_{k\to\infty} d_k(A) \qquad \text{and} \qquad \overline{d}(A) = \limsup_{k\to\infty} d_k(A)$$

and define $d(A)$ to be the common limit if it exists.

The values of $\overline{\rho}$ and $\overline{d}$ may differ, but not by much.

Lemma 2.14 ([8]). For all sets $A \subseteq \mathbb{N}$, we have

$$\frac{\underline{d}(A)}{2} \leq \underline{\rho}(A) \qquad \text{and} \qquad \frac{\overline{d}(A)}{2} \leq \overline{\rho}(A) \leq 2\overline{d}(A).$$

Proof. For all k, we have

$$\begin{aligned} d_k(A) &= \frac{|A \cap J_k|}{2^k} \\ &\leq \frac{|A \upharpoonright (2^{k+1})|}{2^k} \\ &= 2\rho_{2^{k+1}}(A). \end{aligned}$$

Since the numbers $\rho_{2^{k+1}}(A)$ are a subsequence of the numbers $\rho_k(A)$, we have $\limsup_{k\to\infty} \rho_{2^{k+1}}(A) \leq \overline{\rho}(A)$, so taking lim sup on both sides of the inequality above gives

$$\overline{d}(A) \leq 2\overline{\rho}(A).$$

[3]The definition of J_k in [8] is slightly different, namely, $J_k = [2^k - 1, 2^{k+1} - 1) \cap \mathbb{N}$.

For the other direction, let $\varepsilon > 0$ be given, and let N be large enough that $d_j(A) < \overline{d}(A) + \varepsilon$ for all $j > N$. For $j \leq N$ we can use the obvious bound $d_j(A) < \overline{d}(A) + \varepsilon + 1$, giving

$$\begin{aligned}
\rho_{2^{k+1}-1}(A) &= \frac{|\{0\} \cap A| + |J_0 \cap A| + \cdots + |J_k \cap A|}{2^{k+1}} \\
&\leq \frac{1 + |J_0 \cap A| + \cdots + |J_k \cap A|}{2^{k+1}} \\
&= \frac{1 + 2^0 d_0(A) + 2^1 d_1(A) + \cdots + 2^k d_k(A)}{2^{k+1}} \\
&\leq \frac{1 + 2^0(\overline{d}(A) + \varepsilon + 1) + \cdots + 2^N(\overline{d}(A) + \varepsilon + 1)}{2^{k+1}} \\
&\qquad + \frac{2^{N+1}(\overline{d}(A) + \varepsilon) + \cdots + 2^k(\overline{d}(A) + \varepsilon)}{2^{k+1}} \\
&= \frac{2^{N+1}}{2^{k+1}} + \overline{d}(A) + \varepsilon.
\end{aligned}$$

As N is fixed and ε was arbitrary, this gives

$$\limsup_{k \to \infty} \rho_{2^{k+1}}(A) \leq \overline{d}(A). \tag{1}$$

Now, if $2^k \leq n < 2^{k+1}$, then

$$\begin{aligned}
\rho_n(A) &= \frac{|A \upharpoonright n|}{n} \\
&= \frac{2^{k+1}}{n} \cdot \frac{|A \upharpoonright n|}{2^{k+1}} \\
&\leq 2 \cdot \frac{|A \upharpoonright n|}{2^{k+1}} \\
&\leq 2 \cdot \frac{|A \upharpoonright (2^{k+1})|}{2^{k+1}} \\
&= 2\rho_{2^{k+1}}(A).
\end{aligned}$$

Taking lim sup on both sides and combining with inequality 1, we get

$$\overline{\rho}(A) \leq 2\overline{d}(A).$$

Let $\varepsilon > 0$ be given. Analogously to the argument that showed inequality 1, we can let N be so that $d_j(A) > \underline{d}(A) - \varepsilon$ for all $j > N$, giving

$$\rho_{2^{k+1}}(A) \geq -\frac{2^{N+1}}{2^{k+1}} + \underline{d}(A) - \varepsilon.$$

If $2^k \le n < 2^{k+1}$, then

$$\begin{aligned}\rho_n(A) &= \frac{|A \restriction n|}{n} \\ &\ge \frac{|A \restriction (2^k)|}{n} \\ &= \frac{2^k}{n}\rho_{2^k}(A) \\ &\ge \frac{1}{2}\rho_{2^k}(A),\end{aligned}$$

which, combining both inequalities and taking lim inf, gives

$$2\underline{\rho}(A) = 2\liminf_{n\to\infty}\rho_n(A) \ge \liminf_{k\to\infty}\rho_{2^k}(A) \ge \liminf_{k\to\infty} d_k(A) = \underline{d}(A). \qquad \blacksquare$$

It is false that $\underline{\rho}(A) \le 2\underline{d}(A)$. If there are infinitely many k such that $J_k \cap A = \emptyset$, then $\underline{d}(A) = 0$, but if the k are sufficiently spaced out and $A(n) = 1$ everywhere else, then we can make $\underline{\rho}(A) = \frac{1}{2}$.

The most important application of this lemma is the following equivalence.

Corollary 2.15. Let $A \subseteq \mathbb{N}$. The following are equivalent.

1. $\rho(A) = 0$
2. $\overline{\rho}(A) = 0$
3. $d(A) = 0$
4. $\overline{d}(A) = 0$
5. $\rho(\overline{A}) = 1$
6. $\underline{\rho}(\overline{A}) = 1$
7. $d(\overline{A}) = 1$
8. $\underline{d}(\overline{A}) = 1$

Proof. If $\overline{d}(A) = 0$, then $\overline{\rho}(A) \le 2\overline{d}(A) = 0$. Conversely, if $\overline{\rho}(A) = 0$, then $\overline{d}(A) \le 2\overline{\rho}(A) = 0$, so 2 and 4 are equivalent.

The equivalence between 1 and 2, and between 3 and 4, is obvious.

Finally, the fact that $d_k(\overline{A}) = 1 - d_k(A)$ and $\rho_n(\overline{A}) = 1 - \rho_n(A)$ directly shows that 1 and 5, 2 and 6, 3 and 7, and 4 and 8 are equivalent. $\blacksquare$

2.4 Asymptotic Notions of Computability

In this section we will explore some relaxations of the notion of computability. Traditionally, to consider that a Turing machine M solves a certain problem, we demand for $M(n)$ to be defined and the correct answer for all inputs n. We will relax this restriction to require $M(n)$ to be defined and correct only for densely many n. The various definitions will vary on how $M(n)$ is required to behave for the other n.

2.4.1 Generic Computability

In this paper we will mostly be concerned with the case where $M(n)$ may not converge, but must be correct where defined.

Definition 2.16. A *generic description* of a function f is a partial function g whose domain has density 1 and $g(n) = f(n)$ wherever g is defined. If g is partial computable, we say that f is *generically computable.* A set $A \subseteq \mathbb{N}$ is generically computable if its characteristic function is generically computable.

We will frequently identify a set with its indicator function; for example, we will say that "f is a generic description of A" when we mean that f is a generic description of the indicator function of A.

Example 2.17. We claim that no 1-random set is generically computable. Let g be any partial computable function with dense domain. In particular, the domain of g is infinite. Let $(a_0, b_0), (a_1, b_1), \ldots$ be a computable enumeration of the graph of g (so that $g(a_i) = b_i$ for all i), and define

$$V_k = \{A \mid A(a_i) = b_i \text{ for all } i \leq k\}.$$

Each V_k is a Σ^0_1 class of measure 2^{-k}, and $\bigcap_{k \in \mathbb{N}} V_k$ is the collection of all sets A whose characteristic function agrees with g where g is defined. Thus, every 1-random set disagrees with g somewhere. Since g was arbitrary, no 1-random set is generically computable. □

Example 2.18. For any set $A \subseteq \mathbb{N}$, let $B = \{2^n \mid n \in A\}$. The set B is a "sparsification" of the set A; they are Turing-equivalent. Nevertheless, the function f such that $f(n) = 0$ if n is not a power of 2, and $f(n)\uparrow$ otherwise, is (partial) computable and a generic description of B. Hence the set B will always be generically computable, regardless of A. This shows that every Turing degree contains a generically computable set. □

Example 2.19. Let $A \subseteq \mathbb{N}$ be a noncomputable set, and define $B = \bigcup_{k \in A} J_k$, where J_k is as in Definition 2.13. Suppose that B were generically computable, and let f be a witness (a generic description of B which is partial computable). By Proposition 2.15, for all sufficiently large k, the function f is defined for most $n \in J_k$. So, to compute whether $k \in A$ or not, just try to compute $f(n)$ for all $n \in J_k$, in an interleaved fashion; it must be defined for at least one $n \in J_k$, and as soon as $f(n)$ is calculated for this n, we know that this is the value of $A(k)$.

That would make A computable, a contradiction. Hence every noncomputable set is Turing-equivalent to a set which is not generically computable. □

2.4.2 Coarse Computability

An alternative notion arises if we demand for $M(n)$ to always be defined, but allow the answer to be wrong sometimes.

Definition 2.20. A *coarse description* of a total function f is a total function g such that the set $\{n \mid f(n) = g(n)\}$ has density 1. If g is computable, we say that f is *coarsely computable.* A set $A \subseteq \mathbb{N}$ is coarsely computable if its characteristic function is coarsely computable.

Identifying sets with their characteristic functions, we will say that a set B is a coarse description of a set A if $\mathbb{1}_B$ is a coarse description of $\mathbb{1}_A$, or equivalently, if the symmetric difference $A \triangle B$ has density 1.

Example 2.21. If A and B are as in Example 2.18, then the empty set is a coarse description of B, so all sets are Turing-equivalent to a coarsely computable set.

If A and B are in Example 2.19 and B is coarsely computable, let C be computable such that $B \triangle C$ is sparse. Then for all sufficiently large k we have $d_k(B \triangle C) < \frac{1}{3}$, so either $d_k(C) < \frac{1}{3}$ (in which case $k \notin A$) or $d_k(C) > \frac{2}{3}$ (in which case $k \in A$). Hence C computes A. So every noncomputable set is Turing-equivalent to a non-coarsely computable set. □

Example 2.22. We can construct a set which is coarsely computable, but not generically computable, through diagonalization, as follows. Let R_e be the set

$$R_e = \{n \in \mathbb{N} : 2^e \mid n \wedge 2^{e+1} \nmid n\}.$$

So R_0 is the set of odd numbers, R_1 is the set of even numbers not divisible by four, R_2 is the set of multiples of four which are not multiples of eight, and so on. Note that $\rho(R_i) = 2^{-e-1}$. We will construct a non-generically computable c.e. set A such that $|A \cap R_e| \leq 1$ for all e. So, for each e, all but finitely many elements of A are contained in R_e, so $\rho(A) = 0$, whence the computable set $\emptyset$ is a coarse description of A.

To achieve this, for each e, simulate the eth Turing machine on every number $n \in R_e$, and wait for it to halt in any of those inputs. If it halts on n, set $n \in A$ if the machine rejected it, and leave $n \notin A$ otherwise, and stop simulating the eth Turing machine (we are done with it). This finishes the construction.

Clearly A is c.e., and if the eth machine halts in some $n \in R_e$, by construction the function it is computing cannot be a generic description of A (as it is getting the value of the nth input wrong). But if the machine never halts for any $n \in R_e$, then the density of its domain is at most $1 - 2^{-e-1}$, so it is also not a generic description. Hence A is coarsely computable but not generically computable. □

Example 2.23. Conversely, we can construct a c.e. set A which is generically computable, but not coarsely computable. Recall that the sets $J_k = [2^k, 2^{k+1}) \cap \mathbb{N}$ form a partition of $\mathbb{N} \setminus \{0\}$. We will have a countable collection of strategies S_e, one for each Turing machine e. S_e will try to prevent the eth Turing machine from coarsely computing A.

We will construct a partial function f (which will be a generic description of $\mathbb{1}_A$) by stages. Strategy S_e starts acting on stage s, and "claims" the smallest unclaimed J_k for itself. If n is one of the last $2^k - 2^{k-e}$ elements of J_k, we define $f(n) = 0$. Otherwise, S_e simulates the eth Turing machine on n for s steps. If the Turing machine does not halt on all the remaining elements, S_e does nothing, and will try again on the next stage. Otherwise, define $f(n)$ to be the opposite value of what the Turing machine answered, and claim the next smallest unclaimed J_k for the next stage. Finally, let $A = f^{-1}(1)$. This finishes the construction.

If strategy S_e claims only finitely many intervals J_k, it means that the eth Turing machine fails to halt in some element of the last claimed interval. Hence the eth Turing machine cannot coarsely compute A.

If strategy S_e claims infinitely many intervals J_k, let B be the set computed by the eth Turing machine. If n is one of the first 2^{k-e} elements of J_k, by construction we have $A(n) = f(n) \neq B(n)$, so $d_k(A \triangle B) \leq 1-2^{-e}$. Since there are infinitely many such k, the set B is not a coarse description of A.

This shows that A is not coarsely computable. To show that A is generically computable, note that if S_e claims infinitely many intervals J_k, then $f(n)\downarrow = A(n)$ for all n in each of these J_k; and if S_e claims only finitely many intervals, it may leave $f(n)$ undefined for up to 2^{k-e} elements in J_k if it is the last claimed interval, but after that S_e stops claiming intervals at all. Hence if D is the domain of f, then $d_k(D) < 1 - 2^{-e}$ for finitely many k. So D is dense, which shows that f is a partial computable generic description of A. □

The following proposition will be useful in Section 3.2.

Proposition 2.24. *If the set X is Church-stochastic, then X is not coarsely computable.*

Proof. Let A be any computable set. We will show that A is not a coarse description of X.

Because $\rho(A \cup \overline{A}) = 1$, either $\overline{\rho}(A) > 0$ or $\overline{\rho}(\overline{A}) > 0$; without loss of generality, assume the former. Let $f : \mathbb{N} \to \mathbb{N}$ be the increasing function which satisfies

$$A = \{f(0), f(1), f(2), \dots\}.$$

Then f is a computable selection function, and because X is Church-stochastic,

$$\lim_{n\to\infty} \frac{|\{k < n \mid f(k) \in X\}|}{n} = \frac{1}{2}.$$

Let $\varepsilon > 0$ be given, and take N large enough that if $n \geq N$ then the limit above is within ε of $\frac{1}{2}$. Further, take some $n > f(N)$ for which $\rho_n(A) > \overline{\rho}(A) - \varepsilon$, let m be the largest integer for which $f(m) < n$, and note $m \geq N$. Then

$$\begin{aligned}\rho_n(A \setminus X) &= \frac{|\{k < n \mid k \in A \wedge k \notin X\}|}{n} \\ &= \frac{|\{j \leq m \mid f(j) \in A \wedge f(j) \notin X\}|}{n} \\ &= \frac{|\{j \leq m \mid f(j) \notin X\}|}{n} \\ &\geq \left(\frac{1}{2} - \varepsilon\right) \frac{m+1}{n} \\ &= \left(\frac{1}{2} - \varepsilon\right) \frac{|\{k < n \mid k \in A\}|}{n} \\ &\geq \left(\frac{1}{2} - \varepsilon\right) (\overline{\rho}(A) - \varepsilon).\end{aligned}$$

Because $\varepsilon > 0$ was arbitrary, this means that $\overline{\rho}(A \setminus X) \geq \overline{\rho}(A)/2$. So A and X disagree on a set with positive upper density, which means that A cannot be a coarse description of X. ■

2.4.3 Dense and Effectively Dense Computability

Generic computability arises when the Turing machine M is allowed to not halt sometimes, and coarse computability arises when M is allowed to make a few mistakes. If both of these relaxations are taken together, we get dense computability.

Definition 2.25. A *dense description* of a function f is a function g such that the set

$$\{n \mid g(n)\downarrow \wedge g(n) = f(n)\}$$

has density 1. If g is partial computable, we say that f is *densely computable*. A set is densely computable if its characteristic function is densely computable.

Clearly all generically computable sets and all coarsely computable sets are densely computable. It is interesting to note, however, that there are sets which are neither generic nor coarsely computable, but which are densely computable. In a sense, this whole is larger than the sum of its two parts.

Example 2.26. Let A be the set constructed in Example 2.23 and B be the set constructed in Example 2.22. Then the join $A \oplus B$ of A and B is neither generically nor coarsely computable, but it is densely computable. □

If we disallow both relaxations (we still demand the Turing machine to always halt, and we forbid it from giving the wrong answer), we can still construct an asymptotic notion of computability which is weaker than just being computable. We allow the machine to return the special symbol □, which means that the machine does not know the answer.

Definition 2.27. An *effective dense description* of a function $f : \mathbb{N} \to \mathbb{N}$ is a total function $g : \mathbb{N} \to \mathbb{N} \cup \{\Box\}$ such that, for all n, we have either $f(n) = g(n)$ or $f(n) = \Box$, and the set

$$\{n \mid g(n) \neq \Box\}$$

has density 1. If g is computable, then f is *effectively densely computable.* A set is effectively densely computable if its characteristic function is effectively densely computable.

Clearly, all effectively densely computable sets are also generically computable and coarsely computable, and a modification of Example 2.18 shows that there are effectively densely computable sets which are not computable.

Example 2.28. We will modify Example 2.23 to construct a set which both generically and coarsely computable, but not effectively densely computable.

Strategy S_e will again simulate the eth Turing machine on the beginning of its claimed intervals. Suppose S_e claimed interval J_k, and that the eth Turing machine computes the function f_e. First we again define $A(n) = 0$ for all of the last $2^k - 2^{k-e}$ elements n of J_k. Once the machine halts on all of the 2^{k-e} starting elements of J_k, if $f_e(n) \neq \Box$ for any of those n, we can define $A(n) = 1 - f_e(n)$ for the smallest one, $A(n) = 0$ for the rest, and S_e can stop claiming intervals. Otherwise, we have $f_e(n) = \Box$ for all of the 2^{k-e} starting elements of J_k; define $A(n) = 0$ for all $n \in J_k$ and claim another interval. This finishes the construction.

If strategy S_e claims infinitely many intervals, it means that $f_e(n) = \Box$ in all the beginning 2^{k-e} elements of J_k for infinitely many k, so f_e is not an effective dense description of A. If strategy S_e claims only finitely many intervals, it means that f_e is undefined in some of the claimed intervals, which thus mean that the eth Turing machine does not effectively densely compute A. Therefore, A is not effectively densely computable.

Finally, the same reasoning as in Example 2.23 shows that A is indeed generically computable; and $A(n) = 1$ for at most one $n \in J_k$, so $\emptyset$ is a coarse description of A, which shows that A is also coarsely computable. $\Box$

The following proposition will be useful in Section 3.3.

Proposition 2.29. *If the set X is Church-stochastic, then X is not densely computable.*

Proof. Let $g : D \subseteq \mathbb{N} \to \{0, 1\}$ be a partial computable function with dense domain. We will show that g is not a dense description of X.

Because $\rho(D) = 1$, at least one of the sets

$$\{n \mid g(n)\downarrow = 1\} \quad \text{and} \quad \{n \mid g(n)\downarrow = 0\}$$

has positive upper density. Assume it is the former without loss of generality.

Let $q \in \mathbb{Q}$ satisfy

$$\frac{q}{2} \leq \overline{\rho}(\{n \mid g(n)\downarrow = 1\}) \leq q.$$

Define the computable set B as follows.

Start with $k = 0$. On stage s, evaluate $g(n)$ for $n \in J_k, J_{k+1}, \ldots$ and wait until, for some $i \geq k$, we have

$$|\{n \in J_i \mid g(n)\downarrow = 1\}| > q2^{i-2}.$$

Because $\overline{\rho}(\{n \mid g(n)\downarrow = 1\}) > q/2$, by Lemma 2.14, we have $\overline{d}(\{n \mid g(n)\downarrow = 1\}) > q/4$, so for each fixed k there is some $i \geq k$ for which the condition above is satisfied. Then define B on J_i to be the first $q2^{i-2}$ elements of $\{n \in J_i \mid g(n)\downarrow = 1\}$, and $B(n) = 0$ if $n \in J_k \cup J_{k+1} \cup \cdots \cup J_{i-1}$. Set $k = i$ and go to the next stage.

Again by Lemma 2.14, this means that $\overline{\rho}(B) > q/8$, so B has a positive upper density. The argument now follows the proof of Theorem 2.24 to show that B and X disagree in a set of positive upper density. Therefore g is not a dense description of X. ■

2.5 Enumeration Operators and Reducibilities

In order to talk about generic degrees, we have to define "generic reduction" between two sets, but simply using Turing reductions will not work.

Generic descriptions are functions. To use functions as oracles, we will consider their graphs; formally, we let $\langle \cdot, \cdot \rangle : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be any fixed pairing function (say, Cantor's pairing function), and define

$$\mathrm{graph}(f) = \{\langle x, f(x) \rangle \mid x \in \mathrm{dom}(f)\}.$$

The problem of using $\mathrm{graph}(f)$ directly as an oracle for a Turing machine is that the machine can query whether a certain number is in the domain of f or not. For example, for any set A, we can define $f(2^k) = 0$ if $k \in A$, leave $f(2^k)$ undefined if $k \notin A$, and set $f(n) = 0$ if n is not a power of 2. Then f is a generic description of the empty set, but it is easy to see that $\mathrm{graph}(f) \geq_{\mathrm{T}} A$. Hence for every set A, there is a generic description of $\emptyset$ which computes it.

The issue is that the function directly queries whether a pair is in $\mathrm{graph}(f)$ or not. To prevent this, we will use enumeration operators instead.

Definition 2.30. Let $W \subseteq \mathbb{N}^* \times \mathbb{N}$ be a set of pairs (F, k), where $k \in \mathbb{N}$ and F is a finite subset of $\mathbb{N}$. Then W is an *enumeration operator* if the set of codes for the elements in W is a c.e. set. (We will identify W with the set of codes.) For any set A, we define W^A by

$$W^A = \{k \mid (F, k) \subseteq W \text{ for some } F \subseteq A\},$$

and we say that W^A is enumeration reducible to A.

The name comes from the idea of a program that can transform any enumeration of A into an enumeration of a set B. Since the elements of A may be presented out of order, all the algorithm ever sees are finite subsets $F \subseteq A$. If the algorithm decides to enumerate k into B, we place the pair (F, k) into W. The result is that $B = W^A$.

As a direct consequence of this definition, if A is a c.e. set, then any W^A is also c.e. for any enumeration operator W.

Proposition 2.31. *For infinite sets, the set W may be taken to be computable; that is, for every enumeration operator W there exists a computable enumeration operator V such that $W^A = V^A$ for all infinite sets A.*

Proof. To decide whether $(F, k) \in V$ or not, let M be a Turing machine which recognizes W and simulate M on all pairs (F', k) with $F' \subseteq F$ for $|F|$ steps. Place $(F, k) \in V$ if M accepts any pair, and define $(F, k) \notin V$ otherwise. This makes V computable.

For any k, if $k \in V^A$, it means that $(F, k) \in V$ for some $F \subseteq A$, which means that $(F', k) \in W$ for some $F' \subseteq F$. Hence $(F', k) \in W$ for some $F' \subseteq A$, so $k \in W^A$.

Conversely, if A is infinite and $k \in W^A$, this means that $(F, k) \in W$ for some $F \subseteq A$. Suppose it takes s steps for M to accept (F, k). Let $S \subseteq A$ be any subset of A with s elements. Then, by construction, $(F \cup S, k) \in V$, and $F \cup S \subseteq A$. Therefore, $k \in V^A$. ■

Proposition 2.32. *The set W may be assumed to be monotonic; that is, for every enumeration operator W there exists an enumeration operator V such that $W^A = V^A$ for all sets A, and if $(F, k) \in V$ then $(F', k) \in V$ for all $F' \supset F$.*

Proof. Define

$$V = \{(F, k) \mid \exists F' \subseteq F[(F', k) \in W]\}.$$

As $V \supseteq W$, we have $W^A \subseteq V^A$ for all A. Furthermore, if $k \in V^A$, then $(F, k) \in V$ for some $F \subseteq A$, so $(F', k) \in W$ for some $F' \subseteq F$, so $(F', k) \in W$ for some $F' \subseteq A$, which means $k \in W^A$. Hence $V^A = W^A$. ■

An observation which will be important below is that we may compose enumeration operators.

Proposition 2.33. *For any enumeration operators V and W there exists an enumeration operator U such that $U^A = V^{W^A}$ for all sets A.*

Proof. Replacing V and W if needed, we may assume that they are monotonic. Define U via

$$U = \Big\{(F, k) \,\Big|\, \exists F' \big[(F', k) \in V \wedge \forall l \in F'[(F, l) \in W]\big]\Big\}$$

U is an enumeration operator because all of the candidate F' must be finite.

Let $k \in V^{W^A}$. Then $(F, k) \in W$ for some set $F \subseteq W^A$. For each $l \in F$ we have $l \in W^A$, so there exists some $F'_l \subseteq A$ such that $(F'_l, l) \in W$. Define $F' = \bigcup_{l \in F} F'_l$. Because W is monotonic, we also have $(F', l) \in W$ for all $l \in F$. Then $(F, k) \in U$, by construction, so $k \in U^A$.

Conversely, let $k \in U^A$, so $(F, k) \in U$ for some $F \subseteq A$. By definition of U, this means that $(F', k) \in V$ for some F' such that $(F, l) \in W$ for all $l \in F'$. Because $F \subseteq A$, this means that $l \in W^A$ for all $l \in F'$, so $F' \subseteq W^A$. Hence $k \in V^{W^A}$. ■

2.5.1 Dense and Generic Reducibility

Definition 2.34. A set A is *nonuniformly generically reducible* to a set B, denoted by $A \leq_{\mathrm{ng}} B$, if for every generic description f of A, there exists a generic description g of B such that f is enumeration reducible to g.

Definition 2.35. A is *uniformly generically reducible* to B, denoted by $A \leq_{\mathrm{ug}} B$, if there is a single enumeration operator W such that W^g is a generic description of A for all generic descriptions g of B.

A direct consequence of Proposition 2.33 is that both $\leq_{\mathrm{ng}}$ and $\leq_{\mathrm{ug}}$ are transitive, and therefore the definitions below make sense.

Definition 2.36. The *uniform* (resp. *nonuniform*) *generic degree* of a set A is the collection of sets B such that $A \leq_{\mathrm{ug}} B$ and $B \leq_{\mathrm{ug}} A$ (resp. $A \leq_{\mathrm{ng}} B$ and $B \leq_{\mathrm{ng}} A$).

For example, the class of generically computable sets is the generic degree of $\emptyset$. In this case, the uniform and the nonuniform generic degrees coincide, but this is not the case in general.

Example 2.37. Let A be any set, and define the sets $\mathcal{R}(A)$ and $\tilde{\mathcal{R}}(A)$ by [2, Section 5]

$$\mathcal{R}(A) = \bigcup_{k \in A} R_k$$

and

$$\tilde{\mathcal{R}}(A) = \bigcup_{k \in A} J_k.$$

Clearly, $A \geq_{\mathrm{T}} \mathcal{R}(A)$ and $A \geq_{\mathrm{T}} \tilde{\mathcal{R}}(A)$.

Because each R_k has positive density, if f is any generic description of $\mathcal{R}(A)$, then $f(n)$ is defined for some $n \in R_k$. Since generic descriptions must be correct wherever defined, we can uniformly reconstruct A from any generic description of $\mathcal{R}(A)$ by simply waiting for some $n \in R_k$ to be in the domain of f. Because $A \geq_{\mathrm{T}} \tilde{\mathcal{R}}(A)$, this means that we can compute $\tilde{\mathcal{R}}(A)$ from any generic description of $\mathcal{R}(A)$, whence

$$\tilde{\mathcal{R}}(A) \leq_{\mathrm{ug}} \mathcal{R}(A).$$

If f is any generic description of $\tilde{\mathcal{R}}(A)$, then for all sufficiently large k, we have that $f(n)$ is defined for at least one $n \in J_k$. Hence we can nonuniformly compute $\mathcal{R}(A)$ from f by "memorizing" the values of $A(k)$ for the finitely many non-sufficiently large k, and just waiting for $f(n)$ to be defined for some $n \in J_k$ if k is sufficiently large. Thus

$$\mathcal{R}(A) \leq_{\text{ng}} \tilde{\mathcal{R}}(A).$$

Finally, choose A to be a non-autoreducible set [12, Exercise 7.3.8] (that is, there is no Turing reduction Φ such that $A(k) = \Phi^{A \setminus \{k\}}(k)$ for all k). We claim that

$$\mathcal{R}(A) \not\leq_{\text{ug}} \tilde{\mathcal{R}}(A)$$

in this case. Indeed, suppose that there exists an enumeration operator W which witnesses $\mathcal{R}(A) \leq_{\text{ug}} \tilde{\mathcal{R}}(A)$. For any k, define $f(n) = A(j)$ if $n \in J_j$ and $j \neq k$, and let $f(n)$ undefined otherwise. Because f is a generic description of $\tilde{\mathcal{R}}(A)$, we must have $W^f(n)\downarrow = A(k)$ for some $n \in R_k$. As f can be uniformly computed from $A \setminus \{k\}$, this shows that $A(k)$ can be uniformly computed from $A \setminus \{k\}$, which shows that A is autoreducible, a contradiction.

Hence the sets $\mathcal{R}(A)$ and $\tilde{\mathcal{R}}(A)$ are nonuniformly generically equivalent, but not uniformly generically equivalent. □

We will use a similar definition for dense reducibility.

Definition 2.38. A set A is *nonuniformly densely reducible* to a set B, denoted by $A \leq_{\text{nd}} B$, if for every dense description f of A, there exists a dense description g of B such that f is enumeration reducible to g. A is *uniformly generically reducible* to B, denoted by $A \leq_{\text{ud}} B$, if there is a single enumeration operator W such that W^g is a dense description of A for all dense descriptions g of B.

Again, Proposition 2.33 guarantees that the definition below makes sense.

Definition 2.39. The *uniform* (resp. *nonuniform*) *dense degree* of a set A is the collection of sets B such that $A \leq_{\text{ud}} B$ and $B \leq_{\text{ud}} A$ (resp. $A \leq_{\text{nd}} B$ and $B \leq_{\text{nd}} A$).

The distinction between uniform and nonuniform is again necessary; see [2, Corollary 5.8] for an example of a pair of sets which are nonuniformly densely equivalent, but not uniformly densely equivalent.

2.5.2 Coarse and Effectively Dense Reducibilities

Coarse and effectively dense descriptions of sets must be defined everywhere, so there is no need to use enumeration operators.

Definition 2.40. A set A is *nonuniformly coarsely reducible* to a set B, denoted by $A \leq_{\text{nc}} B$, if every coarse description of B computes a coarse description of A. The set A is *uniformly coarsely reducible* to B, denoted by $A \leq_{\text{uc}} B$, if there is a single Turing functional Φ such that Φ^C is a coarse description of A for all coarse descriptions C of B.

The definition of uniform and nonuniform coarse degrees is analogous to the previous definitions. See [7, Theorem 2.6] for an example of two nonuniformly coarsely equivalent sets which are not uniformly coarsely equivalent.

The definition for effectively dense degrees is analogous.

Definition 2.41. A set A is *nonuniformly effectively densely reducible* to a set B, denoted by $A \leq_{\mathrm{ned}} B$, if every effectively dense description of B computes an effectively dense description of A. The set A is *uniformly effectively densely reducible* to B, denoted by $A \leq_{\mathrm{ued}} B$, if there is a single Turing functional Φ such that Φ^C is an effectively dense description of A for all effectively dense descriptions C of B.

The definition of uniform and nonuniform effectively degrees is analogous to the previous definitions. See [2, Corollary 5.3] for a pair of sets which are nonuniformly effectively densely equivalent, but which are not uniformly effectively densely equivalent.

3 Minimal Pairs and Randomness

Informally speaking, if A and B are any two sets, we can ask whether they have some common "computational power". Clearly both A and B can compute any computable set, but is there anything more? Perhaps surprisingly, there are many pairs without any common computational power (besides the computable sets), which we call "minimal pairs".

We start with the Turing degrees.

3.1 Minimal Pairs in the Turing Degrees

Definition 3.1. The *upper cone* above a set A is the collection

$$\{B \mid B \geq_{\mathrm{T}} A\},$$

and the *lower cone* below A is the collection

$$\{B \mid B \leq_{\mathrm{T}} A\}.$$

The intersection of these two cones is the *Turing degree* of A.

For any two sets A and B, it is easy to see that their upper cones intersect; in fact, the intersection of the upper cones above A and B is exactly the upper cone above $A \oplus B$.

Trivially, the lower cones also intersect, because all lower cones contain the computable sets. And, in some cases, their intersection contain *only* the computable sets.

Definition 3.2. A *minimal pair* for the Turing degrees is a pair of sets (A, B) such that neither A nor B are computable, but if $C \leq_{\mathrm{T}} A$ and $C \leq_{\mathrm{T}} B$, then C is computable.

That is, the intersection of the lower cones below A and B is the class of computable sets.

We will provide a proof of existence of minimal pairs based on the Lebesgue Density Theorem, using a "majority vote" argument.

Lemma 3.3. If A is noncomputable, then the upper cone above A has measure zero.

Proof. Let V be the upper cone above A, and assume it has positive measure.

For each e, define $V_e = \{B \mid \Phi_e^B = A\}$; that is, the class V_e contains all the oracles that e can use to compute A. Note that V is the union of all V_e, so if $\mu(V) > 0$ then $\mu(V_e) > 0$ for some e.

By the Lebesgue Density Theorem, there is some $\sigma \in 2^{<\omega}$ such that $\mu(V_e \cap [\![\sigma]\!]) > \frac{2}{3}\mu([\![\sigma]\!])$, so we can do a majority vote inside $[\![\sigma]\!]$: given n, compute $\Phi_e^\tau(n)[|\tau|]$ for each $\tau \succcurlyeq \sigma$, until either

- the measure of the τ such that $\Phi_e^\tau(n)[|\tau|]{\downarrow} = 0$ surpasses $\frac{1}{2}\mu([\![\sigma]\!])$, in which case we know that $A(n) = 0$; or
- the measure of the τ such that $\Phi_e^\tau(n)[|\tau|]{\downarrow} = 1$ surpasses $\frac{1}{2}\mu([\![\sigma]\!])$, in which case we know that $A(n) = 1$.

One of these two must happen, and when it does the majority vote is the correct answer.

Therefore, if the measure of V is nonzero, then A is computable. ■

Proposition 3.4. *Each noncomputable set forms a minimal pair with measure-1 many sets.*

Proof. Let A be noncomputable. There are countably many noncomputable sets $D \leq_{\mathrm{T}} A$, and for each of those sets, the upper cone $V_D = \{C \mid C \geq_{\mathrm{T}} D\}$ has measure zero. We claim that if B is noncomputable and not in any of these cones, then (A, B) forms a minimal pair for the Turing degrees.

Indeed, if $C \leq_{\mathrm{T}} A, B$, then clearly $B \in V_C$; by construction, this is only possible if C is computable. So (A, B) is a minimal pair for the Turing degrees.

Since there are countably many $D \leq_{\mathrm{T}} A$, the family of valid values for B has measure 1. ■

Hence, a simple application of Fubini's Theorem shows that the collection of pairs (A, B) which form a minimal pair for the Turing degrees has measure 1. Intuitively, this means that any two sets picked "at random" will be a minimal pair for the Turing degrees. In fact, we can show that a fairly low degree of randomness suffices.

Proposition 3.5. *If A and B are relatively weakly 2-random, then A and B form a minimal pair for the Turing degrees.*

Proof [4, Corollary 8.12.4]. Let C be a noncomputable set such that $C \leq_{\mathrm{T}} B$, and suppose for the sake of contradiction that $C \leq_{\mathrm{T}} A$. Let e satisfy $\Phi_e^A = C$, and define

$$\mathcal{S} = \{X \mid \forall n \exists s [\Phi_e^X(n)[s] = C(n)]\}.$$

Then $\mathcal{S}$ is a $\Pi_2^{0,B}$ class which is contained in the upper cone above C. By Lemma 3.3, the class $\mathcal{S}$ has measure 0. Since A is weak 2-random relative to B, this means that $A \notin \mathcal{S}$, a contradiction. ■

3.2 Minimal Pairs in the Coarse Degrees

For coarse reducibility, we have a similar definition, though we have to pay attention to uniformity.

Definition 3.6. Two sets A and B form a *minimal pair for the uniform coarse degrees* if both are non-coarsely computable and if C is any set such that $C \leq_{\mathrm{uc}} A, B$, then C is coarsely computable.

Being a minimal pair for the nonuniform coarse degrees is defined similarly.

Because the relation $\leq_{\mathrm{uc}}$ has stricter requirements than $\leq_{\mathrm{nc}}$, if two sets form a minimal pair for the nonuniform coarse degrees then they also form a minimal pair for the uniform coarse degrees, so it suffices to show the former.

Lemma 3.7 ([7, Theorem 5.2]). If A is non-coarsely computable and X is weakly 3-random relative to A, then X does not compute any coarse description of A.

Proof. Suppose for the sake of contradiction that $A \leq_{\mathrm{nc}} X$, and let Φ be a Turing functional such that Φ^X is a coarse description of A. Define the class $\mathcal{P}$ by

$$\mathcal{P} = \{Y \mid \Phi^Y \text{ is a coarse description of } A\}.$$

For Y to be in $\mathcal{P}$, it must be the case that Φ^Y is total (which is a Π_2^0 property) and that $\lim_k \rho_k(\Phi^Y \triangle A) = 0$, which we can express as

$$\forall \varepsilon \exists K \forall s, k > K \Big[\big(\forall n < k (\Phi^Y(n)[s]\downarrow)\big) \implies \rho_k(\Phi^Y[s] \triangle A) < \varepsilon \Big],$$

which is a $\Pi_3^{0,A}$ property. Hence $\mathcal{P}$ is a $\Pi_3^{0,A}$ class.

Because X is weakly 3-random relative to A, it is not contained in any $\Pi_3^{0,A}$ class of measure 0; since $X \in \mathcal{P}$, we must have $\mu(\mathcal{P}) > 0$. Using the Lebesgue Density Theorem, there exists some σ such that $\mu(\mathcal{P} \cap [\![\sigma]\!]) > \frac{5}{6} 2^{-|\sigma|}$. By replacing $\Phi^{\tau Y}$ with $\Phi^{\sigma Y}$ for all $|\tau| = |\sigma|$ (except if $\tau = X \upharpoonright |\sigma|$), we may assume that $\mu(\mathcal{P}) > \frac{4}{5}$; that is, for more than $\frac{4}{5}$ of all Y, the set Φ^Y is a coarse description of A. (The exception of not replacing if $\tau = X \upharpoonright |\sigma|$ exists solely to keep $X \in \mathcal{P}$, and it is not really needed in the rest of the argument.)

Define the set D as follows. Recall that $J_k = [2^k, 2^{k+1}) \cap \mathbb{N}$. For each k, find some integer s_k and a finite set S_k of s_k-sized strings such that $\Phi^\sigma(n)\downarrow$ for all $n \in J_k$ and all $\sigma \in S_k$, and $|S_k| > \frac{4}{5} 2^{n_k}$. We know such set exists because $\mu(\mathcal{P}) > \frac{4}{5}$. Then pick a set $R_k \subseteq S_k$ such that $|R_k| > \frac{1}{2} 2^{n_k}$ which minimizes

$$\max_{\sigma,\tau \in R_k} d_k(\Phi^\sigma \triangle \Phi^\tau).$$

Finally, let $D(n) = \Phi^\tau(n)$ for all $n \in J_k$, where τ is some fixed element of R_k (say, the lexicographically least element). We claim that D is a coarse description of A, which implies that A is coarsely computable.

Let $\varepsilon > 0$ be given, and define $\mathcal{B}_k$ by

$$\mathcal{B}_k = \{Y \mid \Phi^Y(n)\downarrow \text{ for all } n \in J_k \text{ and } d_k(\Phi^Y \triangle A) < \varepsilon\}.$$

Note that in the definition of S_k, all s_k-sized prefixes of elements in $\mathcal{B}_k$ are "valid choices".

Suppose that $\mu(\mathcal{B}_k) > \frac{4}{5}$ for some k. This means that at least $\frac{4}{5}$ of all the strings σ of length s_k satisfy $d_k(\Phi^\sigma \triangle A) < \varepsilon$, thus at least $\frac{3}{5} 2^{s_k}$ strings in S_k satisfy $d_k(\Phi^\sigma \triangle A) < \varepsilon$, which means that there is a set $R \subseteq S_k$ with $|R| > \frac{3}{5} 2^{s_k}$ for which

$$\max_{\sigma,\tau \in R} d_k(\Phi^\sigma \triangle \Phi^\tau) < 2\varepsilon.$$

Since R_k minimizes the value above, we know that $d_k(\Phi^\sigma \triangle \Phi^\tau) < 2\varepsilon$ for all $\sigma, \tau \in R_k$.

Since $\mu(\mathcal{B}_k) > \frac{4}{5}$, at least one $\sigma \in R_k$ is a prefix of an element in $\mathcal{B}_k$, so if $\tau \in R_k$ is the string used to define D we have

$$\begin{aligned} d_k(D \triangle A) &= d_k(\Phi^\tau \triangle A) \\ &\le d_k(\Phi^\tau \triangle \Phi^\sigma) + d_k(\Phi^\sigma \triangle A) \\ &\le 2\varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

If $Y \in \mathcal{P}$, by Corollary 2.15 we have $Y \in \mathcal{B}_k$ for all sufficiently large k. Hence the union of the sets $\mathcal{C}_k = \bigcap_{j>k} \mathcal{B}_k$ contains all elements in $\mathcal{P}$. This means that $\mu(\mathcal{C}_k) > \frac{4}{5}$ for all large enough k, which implies that $\mu(\mathcal{B}_k) > \frac{4}{5}$ for all large enough k.

Therefore, $d_k(D \triangle A) < 3\varepsilon$ for all large enough k. Since ε was arbitrary, this shows that D is a coarse description of A, contradicting the hypothesis that A is not coarsely computable. ■

Corollary 3.8. If A and B are relatively weakly 3-random, then they form a minimal pair for relative coarse computability.

Proof. By Propositions 2.5 and 2.24, neither A nor B is coarsely computable, so we have to show that if C is coarsely computable relative to A and B, then C is coarsely computable.

If not, let Y be a coarse description of C which is computable in B. Then A is weakly 3-random relative to Y, so by the lemma above we have Y is not coarsely computable relative to A, which means that X is not coarsely computable relative to A, a contradiction. ■

3.3 Minimal Pairs in the Dense Degrees

For the dense degrees, the definitions are similar.

Definition 3.9. Two sets A and B form a *minimal pair for the uniform dense degrees* if both are non-densely computable and if C is any set such that $C \leq_{\text{uc}} A, B$, then C is densely computable.

A minimal pair for the nonuniform dense degrees is defined similarly.

Because dense computability is more flexible than coarse computability, showing a result similar to Theorem 3.8 will require more work. We start with a lemma [2, Lemma 6.1].

Definition 3.10. If $\mathcal{S} = \{\mathcal{S}_n \subseteq 2^\omega\}_{n\in\mathbb{N}}$ is a sequence of measurable subsets of 2^ω and $A \in 2^\omega$, define $\mathcal{S}(A) = \{n \mid A \in \mathcal{S}_n\}$.

Lemma 3.11. Let $a, b, q \in [0, 1]$ and $\mathcal{S} = \{\mathcal{S}_n\}_{n\in\mathbb{N}}$. Suppose that

$$\overline{\rho}(\{n \mid \mu(\mathcal{S}_n) < q\} > a$$

and

$$\mu(\{A \mid \rho(\mathcal{S}(A)) = 1\}) > b.$$

Then $(1 - q)a + b \leq 1$.

Proof. Fix $r < 1$ and define the class $\mathcal{X}_n$ by

$$\mathcal{X}_n = \{A \mid \forall k > n[\rho_k(\mathcal{S}(A)) > r]\}.$$

The union of all $\mathcal{X}_n$ contains the set $\{A \mid \rho(\mathcal{S}(A)) = 1\}$, and $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}$. Therefore, there exists some N such that $\mu(\mathcal{X}_n) > b$ for all $n > N$. Now fix $n > N$ to satisfy

$$\rho_n(\{k \mid \mu(\mathcal{S}_k) < q\}) > a$$

(such n exist by hypothesis), and consider the equality

$$\frac{1}{n}\sum_{j<n}\int_{2^\omega} \mathbb{1}_{\mathcal{S}_j}\,\mathrm{d}\mu = \frac{1}{n}\int_{2^\omega}\sum_{j<n} \mathbb{1}_{\mathcal{S}_j}\,\mathrm{d}\mu. \tag{2}$$

This equality is true because integrals commute with finite sums. We will provide an upper bound for the left side of the equality and a lower bound for the right side of the equality which, together, will give us the result.

For the right side, we will compare it with $\rho_n(\{n \mid \mu(\mathcal{S}_n) < q\})$; we have

$$\frac{1}{n}\sum_{j<n}\int_{2^\omega} \mathbb{1}_{\mathcal{S}_j}\,\mathrm{d}\mu = \frac{1}{n}\sum_{j<n}\mu(\mathcal{S}_j).$$

Due to the choice of n, at least an of the $\mathcal{S}_j$ satisfy $\mu(\mathcal{S}_j) < q$. For the remaining $n - an$ of them, we can simply bound $\mu(\mathcal{S}_j)$ by 1. This gives

$$\begin{aligned}\frac{1}{n}\sum_{j<n}\int_{2^\omega} \mathbb{1}_{\mathcal{S}_j}\,\mathrm{d}\mu &\leq \frac{anq + (n - an)}{n} \\ &= 1 - (1 - q)a.\end{aligned}$$

For the left side, we can restrict the integration domain to get

$$\frac{1}{n}\int_{2^\omega}\sum_{j<n}\mathbb{1}_{\mathcal{S}_j}\mathrm{d}\mu \geq \frac{1}{n}\int_{\mathcal{X}_n}\sum_{j<n}\mathbb{1}_{\mathcal{S}_j}\mathrm{d}\mu.$$

Because $n > N$, if $A \in \mathcal{X}_n$ we have $\rho_n(\mathcal{S}(A)) > r$, which expands to

$$\sum_{j<n}\mathbb{1}_{S_j}(A) > rn.$$

Substituting into the integral gives

$$\begin{aligned}\frac{1}{n}\int_{2^\omega}\sum_{j<n}\mathbb{1}_{\mathcal{S}_j}\mathrm{d}\mu &\geq \frac{1}{n}\int_{\mathcal{X}_n} rn\mathrm{d}\mu \\ &= r\mu(\mathcal{X}_n) > rb.\end{aligned}$$

Combining both inequalities gives $rb < 1-(1-q)a$, which rearranges to $(1-q)a + rb < 1$. Because $r < 1$ was arbitrary, we finish the proof. ∎

The lemma above will be used to prove the following theorem, which will enable us to implement the "majority vote" argument for dense degrees.

Theorem 3.12. Let $\mathcal{S} = \{\mathcal{S}_n \subseteq 2^\omega\}_{n\in\mathbb{N}}$ and q satisfy

$$\mu(\{A \mid \rho(\mathcal{S}(A)) = 1\}) > q.$$

Then

$$\rho(\{n \mid \mu(\mathcal{S}_n) \geq q\}) = 1.$$

Proof. The result is trivial if $q = 0$, so assume $q > 0$. Define p by

$$p = \overline{\rho}(\{n \mid \mu(\mathcal{S}_n) < q\}),$$

and assume for the sake of contradiction that $p > 0$. Let $\varepsilon > 0$ be such that $\frac{1}{p\varepsilon} \in \mathbb{N}$ and

$$\mu(\{A \mid \rho(\mathcal{S}(A)) = 1\} > q + \varepsilon.$$

We will construct a subsequence $\mathcal{T}$ of $\mathcal{S}$ such that setting $a = 1-\varepsilon$ and $b = q+\varepsilon$ then $a, b, q, \mathcal{T}$ satisfy the hypothesis of the lemma above, which is a contradiction because $a(1-q) + b = 1 + q\varepsilon$ in this case. Thus $p = 0$.

Define the set $I = \{n_0 < n_1 < \cdots\}$ to be the collection of all n for which either $n \equiv 0 \pmod{\frac{1}{p\varepsilon}}$ or $\mu(\mathcal{S}_n) < q$, and define

$$\mathcal{T}_j = \mathcal{S}_{n_j}.$$

Observe that $\underline{\rho}(I) \geq p\varepsilon$, because I contains all multiples of $\frac{1}{p\varepsilon}$, and by definition of p we have $\overline{\rho}(I) \leq p\varepsilon + p$. Intuitively, $\mathcal{T}$ has a positive fraction of all entries of $\mathcal{S}$.

Let $\delta > 0$ be given. There is some N such that $\rho_n(I) < p(1+\varepsilon) + \delta$ for all $n > N$. On the other hand, by definition of p, there are infinitely many n such that

$$\rho_n(\{i \mid \mu(\mathcal{S}_i) < q\}) > p - \delta.$$

If $n > N$ satisfies this condition, then

$$\begin{aligned}
\rho_k(\{n \mid \mu(\mathcal{T}_n) < q\}) &= \rho_k(\{j \mid \mu(\mathcal{S}_{n_j}) < q\}) \\
&= \frac{\rho_{n_k}(I \cap \{n \mid \mu(\mathcal{S}_n) < q\})}{\rho_{n_k}(I)} \\
&\geq \frac{\rho_{n_k}(\{n \mid \mu(\mathcal{S}_n) < q\})}{\rho_{n_k}(I)} \\
&> \frac{p-\delta}{p(1+\varepsilon)+\delta}.
\end{aligned}$$

This means that

$$\overline{\rho}(\{n \mid \mu(\mathcal{T}_n) < q\}) \geq \frac{p-\delta}{p(1+\varepsilon+\delta}.$$

Since $\delta > 0$ was arbitrary, this means that

$$\overline{\rho}(\{n \mid \mu(\mathcal{T}_n) < q\}) \geq \frac{1}{1+\varepsilon} > 1 - \varepsilon.$$

Therefore, a, b and $\mathcal{T}$ satisfy the first condition of Lemma 3.11.

Now, since $\mathcal{T}$ is a subsequence of $\mathcal{S}$, if $\rho(\mathcal{S}(A)) = 1$ then

$$\begin{aligned}
\rho(\mathcal{T}(A)) &= \lim_k \rho_k(\{j \mid A \in \mathcal{S}_{n_j}\}) \\
&= \lim_k \frac{n_k}{k} \rho_{n_k}(\{i \mid a \in \mathcal{S}_i\} \cap I) \\
&= \lim_k \frac{n_k}{k} \rho_{n_k}(\mathcal{S}(A) \cap I) \\
&= \lim_k \frac{\rho_{n_k}(\mathcal{S}(A) \cap I)}{\rho_{n_k}(I)} \\
&= \lim_k \frac{\rho_{n_k}(I) - \rho_{n_k}(I \cap \overline{\mathcal{S}(A)})}{\rho_{n_k}(I)} \\
&= 1 - \lim_k \frac{\rho_{n_k}(I \cap \overline{\mathcal{S}(A)})}{\rho_{n_k}(I)} \\
&\geq 1 - \lim_k \frac{\rho_{n_k}(\overline{\mathcal{S}(A)})}{\rho_{n_k}(I)} \\
&= 1,
\end{aligned}$$

because the numerator goes to zero whereas the denominator stays (asymptotically) between $p\varepsilon$ and $p(1+\varepsilon)$. This means that

$$\mu(\{A \mid \rho(\mathcal{T}(A)) = 1\}) \geq \mu(\{A \mid \rho(\mathcal{S}(A)) = 1\}) > q + \varepsilon.$$

Hence b and $\mathcal{T}$ satisfy the second condition of Lemma 3.11. As noticed before, this means that $1 + q\varepsilon \leq 1$, a contradiction. ■

Now we can show the analogue to Proposition 3.7 for dense reducibility.

Proposition 3.13 ([2, Theorem 6.4]). *If A is non-densely computable and X is weakly 4-random relative to A, then there are no X-computable dense descriptions of A. (In particular, $A \not\leq_{\mathrm{nd}} X$.)*

Proof. Suppose not, and let Φ be a Turing functional for which Φ^X is a dense description of A. Define the class

$$\mathcal{F} = \{Y \mid \Phi^Y \text{ is a dense description of } A\}.$$

Note that $\mathcal{F}$ is a $\Pi^{0,A}_4$ class; indeed,

$$Y \in \mathcal{F} \quad \Leftrightarrow \quad \forall k \exists N \forall n > N \exists s \Big[\big| \{x < n \mid \Phi^Y(x)[s]\downarrow = A(x)\} \big| > n(1 - 2^{-k}) \Big]$$

Since $X \in \mathcal{F}$ and X is weakly 4-random relative to A, it follows that $\mu(\mathcal{F}) > 0$. By replacing Φ if necessary, we may assume that $\mu(\mathcal{F}) > \frac{4}{5}$; indeed, let σ satisfy $\mu(\mathcal{F} \cap [\![\sigma]\!]) > \frac{5}{6} 2^{-|\sigma|}$ (which exists by the Lebesgue Density Theorem) and if $|\tau| = |\sigma|$ replace $\Phi^{\tau Z}$ with $\Phi^{\sigma Z}$, except if $\tau = X \upharpoonright |\sigma|$ (this exception guarantees $X \in \mathcal{F}$).

As before, our goal is to show that A is densely computable, resulting in a contradiction. We will use Theorem 3.12. For each n, define $\mathcal{S}_n = \{Y \mid \Phi^Y(n)\downarrow = A(n)\}$, and $\mathcal{S} = \{\mathcal{S}_n\}_{n \in \mathbb{N}}$. Each $\mathcal{S}_n$ is a $\Sigma^{0,A}_1$ class, and thus measurable. Furthermore, $\mathcal{S}(Y) = \{n \mid \Phi^Y(n)\downarrow = A(n)\}$, so, by definition, $\rho(\mathcal{S}(Y)) = 1$ if and only if Φ^Y is a generic description of A. Hence

$$\mu(\{Y \mid \rho(\mathcal{S}(Y)) = 1\}) > \frac{4}{5}.$$

Therefore, by Theorem 3.12,

$$\rho(\{n \mid \mu(\mathcal{S}_n) \geq \tfrac{4}{5}\}) = 1.$$

We define the partial function g as follows. For $g(n)$, find an i, an s, and a set R of s-sized strings for which $\Phi^\sigma(n)\downarrow = i$ and $|R| > 2^{s-1}$ (that is, more than half of all possible s-sized strings σ agree that $\Phi^\sigma(n)\downarrow = i$) and define $g(n) = i$. Since such set exists for at most one i, this partial function is computable. If $\mu(\mathcal{S}_n) > \frac{4}{5}$ then the set R above exists and $g(n) = A(n)$. Since this happens for densely many n, it follows that g is a dense description of A. Thus A is densely computable. ■

Proposition 3.14. *If X and Y are relatively weakly 4-random, then they form a minimal pair for relative dense computability.*

Proof. By Propositions 2.5 and 2.29, neither X nor Y are densely computable. Let C be a set which is densely computable relative to both X and Y, and let Φ be a Turing functional for which Φ^Y is a dense description of C.

By the Low Basis Theorem [12, Theorem 3.7.2], there exists a completion D of Φ^Y which is low relative to Y. By Theorem 2.11, the set X is weakly 2-random relative to Y'', so it is weakly 2-random relative to D'', so it is weakly 4-random relative to D. This means there are no X-computable dense descriptions of D, which means there are no X-computable dense descriptions of C, which contradicts the hypothesis that C is densely computable relative to C. ■

4 Minimal Pairs for Generic Computability

As we've seen in the previous section, the Turing, coarse, and dense degrees have measure-1 many minimal pairs. The situation for generic computability is more complicated, and is the object of study of this and the next section.

4.1 There are no minimal pairs for relative generic computability

A result from Igusa [9], which predates the results on coarse and dense degrees, states that, for the more restricted notion of relative generic computability, there are actually *no* minimal pairs.

Theorem 4.1. If A and B are noncomputable sets, then there is a set C which is generically computable relative to both A and B, but which is not generically computable.

To obtain the set C, we will construct two total Turing functionals Φ and Ψ such that, for almost all sets X and Y, the sets Φ^X and Ψ^Y are dense, but the union $\Phi^X \cup \Psi^Y$ does not contain any dense c.e. subset.

Under these conditions, we can let $C = \Phi^X \cup \Psi^Y$. The set X can generically compute C by letting $f(n) = 1$ if $\Phi^X(n) = 1$ and leaving $f(n)$ undefined otherwise. Then f is an X-computable function with dense domain which agrees with $\mathbb{1}_C$ wherever it is defined. The set Y similarly generically computes C. And C is not generically computable, because if f agrees with $\mathbb{1}_X$ wherever it is defined, then $f^{-1}(1)$ is a dense subset of X, which thus cannot be c.e.

Definition 4.2. A set X has a *gap of size* 2^{-e} *at* J_i if the last 2^{i-e} elements of J_i are not in X; algebraically, if $X \cap [2^{i+1} - 2^{i-e}, 2^{i+1}) = \emptyset$.

Lemma 4.3. Let X be a set such that all elements missing from it are due to entire gaps being missing. Then X is dense if and only if for each e it only has finitely many gaps of size e.

Proof. Under these constraints, the set X has a gap of size 2^{-e} at J_i if and only if $d_i(X) \leq 1 - 2^{-e}$. The lemma then follows from Corollary 2.15. ■

We first prove the following special case of 4.1.

Theorem 4.4. There exist total Turing functionals Φ and Ψ such that, if A and B are not Δ^0_2 sets, then $\Phi^A \cup \Psi^B$ is dense but contains no dense c.e. set.

Proof. We will construct the total Turing functionals Φ and Ψ in stages. At the beginning of stage s, we will have $\Phi^X(n)$ and $\Psi^Y(n)$ defined for all $n < 2^s$, using only the first s bits of X and Y. (In other words, we will have defined $\Phi^\sigma(n)$ for all $n < 2^s$ and all strings σ with $|\sigma| = s$, and similarly for Ψ.) For definiteness, define $\Phi^X(0) = \Psi^Y(0) = 1$ for all X, Y.

Strategy e starts acting on stage e. Its goal is to either diagonalize against W_e being a subset of any $\Phi^X \cup \Psi^Y$, or, failing that, at least make sure W_e is not a dense set. To do so, on stage s, enumerate W_e for s steps, and define $T_{e,s}$ to be the 4-ary tree of all pairs $\langle\sigma, \tau\rangle$ such that the first $2^{|\sigma|}$ bits of $W_e[s]$ are contained in $\Phi^X \cup \Psi^Y$. Formally,

$$T_{e,s} = \left\{\langle\sigma, \tau\rangle \;\middle|\; k := |\sigma| = |\tau| \le s \wedge W_e[s] \upharpoonright 2^k \subseteq \Phi^\sigma \cup \Psi^\tau\right\}$$

Note that, since the first 2^s bits of Φ^X and Ψ^Y were already defined, the tree $T_{e,s}$ is finite and uniformly computable.

If $T_{e,s}$ has no members $\langle\sigma, \tau\rangle$ with $|\sigma| = |\tau| = s$, then Strategy e succeeded, as $W_e[s] \subseteq W_e$ and for any X, Y, the set $\Phi^X \cup \Psi^Y$ is missing some element of $W_e[s]$.

Otherwise, let $\langle\sigma, \tau\rangle$ be the leftmost pair of $T_{e,s}$ with $|\sigma| = |\tau| = s$. Let $\hat\sigma$ and $\hat\tau$ be the longest prefixes of σ and τ such that the pair $\langle\hat\sigma, \hat\tau\rangle$ is unmarked, mark it, and place a gap of size 2^{-e} at P_s in both $\Phi^{\hat\sigma}$ and $\Psi^{\hat\tau}$. That is, Strategy e demands for both $\Phi^{\hat\sigma}$ and $\Psi^{\hat\tau}$ to have gaps of size 2^{-e} at P_s.

(Intuitively, if T_e is the union of all $T_{e,s}$, we will "sacrifice" the leftmost path of T_e, in the sense that the diagonalization will fail here; but since we are placing gaps of size 2^{-e}, for W_e to be contained in $\Phi^X \cup \Psi^Y$ it must not be dense.)

We essentially let the strategies act independently. On stage s, only the first s strategies acted, placing gaps at P_s for various values of X and Y. Define the remaining values of $\Phi^X(n)$ and $\Psi^Y(n)$ to be 1 (for $n < 2^{s+1}$).

This defines $\Phi^X(n)$ and $\Psi^Y(n)$ for all X and Y and all $n < 2^{s+1}$, so the construction is finished. We now prove correctness.

For each e, let T_e be the union of all $T_{e,s}$ for all s. If, for some s, the tree $T_{e,s}$ has no members of length s, then so will T_e. Therefore, T_e is finite, and Strategy e stopped acting at stage s. This means that W_e is not contained in $\Phi^X \cup \Psi^Y$ for any X, Y.

So suppose that T_e is infinite. Note that, for a pair $\langle\sigma, \tau\rangle$ with $|\sigma| = |\tau| < s$, if $\langle\sigma, \tau\rangle \notin T_{e,s}$, then $\langle\sigma, \tau\rangle \notin T_{e,t}$ for any $t > s$. This implies that the tree T_e is computable. Let X_e, Y_e be the two sets corresponding to the leftmost path in T_e; note that these sets are Δ^0_2.

Since all prefixes $\langle\sigma, \tau\rangle$ of (X_e, Y_e) of length s are contained in $T_{e,s}$, they were chosen infinitely often by Strategy e (because infinitely often they were the leftmost pair of $T_{e,s}$), so all prefixes of X_e and Y_e are marked. Therefore,

$\Phi^{X_e} \cup \Psi^{Y_e}$ has infinitely many gaps of size 2^{-e}. But because $W_e \subseteq \Phi^{X_e} \cup \Psi^{Y_e}$, by the definition of $T_{e,s}$, the set W_e itself has infinitely many gaps of size 2^{-e}. Hence W_e is not dense. This means that, for all X and Y, the set $\Phi^X \cup \Psi^Y$ contains no dense c.e. subset.

Finally, we will show that if, for all e, we have $X \neq X_e$ and $Y \neq Y_e$, then $\Phi^X \cup \Psi^Y$ is dense. Let $\langle \sigma, \tau \rangle$ be a prefix of (X_e, Y_e) which is not a prefix of (X, Y). At some stage, the pair $\langle \sigma, \tau \rangle$ will be marked by Strategy e, and because the pair (X_e, Y_e) is the leftmost path of T_e, at a further stage t, all marked pairs will be extensions of $\langle \sigma, \tau \rangle$. Then, for all $s > t$, Strategy e will never place a gap of size 2^{-e} at P_s in $\Phi^X \cup \Psi^Y$.

Since A and B are not Δ^0_2 and all X_e and Y_e are, the set $\Phi^X \cup \Psi^Y$ is dense (as it has only finitely many gaps of size 2^{-e}), but it contains no dense c.e. set. ■

In the proof above, we constructed two total Turing functionals Φ and Ψ which almost always yield a dense set. The sets X_e and Y_e are the only exceptions. The theorem works for any non-Δ^0_2 set because X_e and Y_e themselves are Δ^0_2.

The sets X_e and Y_e are the leftmost path of a certain computable tree. To prove the theorem for noncomputable Δ^0_2 sets, we will choose different paths.

Proof of 4.1. The case where the noncomputable sets A and B are not Δ^0_2 is covered by Theorem 4.4, so assume A is Δ^0_2 but B is not.

Modify the proof of the theorem above as follows. By the recursion theorem, we may assume we have an index for the tree T_e. Since T_e is a computable 4-ary tree, by the cone-avoidant basis theorem [4, Theorem 2.19.10], there is a Δ^0_2 path Z_e through T_e which does not compute A, and it is possible to uniformly compute a Δ^0_2 index for Z_e. Then, on stage s, Strategy e computes $Z_e[s]$ (an approximation to Z_e) and chooses $\langle \sigma, \tau \rangle$ to be the longest prefix of $Z_e[s]$ which is contained in $T_{e,s}$, instead of taking the leftmost path. The rest of the construction (marking prefixes of $\langle \sigma, \tau \rangle$ and placing gaps) is the same.

By the same arguments as before, if $X \neq X_e$ and $Y \neq Y_e$, then $\Phi^X \cup \Psi^Y$ is dense, and for all X and Y the set $\Phi^X \cup \Psi^Y$ contains no dense c.e. subset.

But here, if T_e is infinite, we will have $Z_e = (X_e, Y_e)$, so neither X_e nor Y_e compute A. Therefore, we have $A \neq X_e$ for all e. Since $B \neq Y_e$ for all e, because B is not Δ^0_2, the set $\Phi^A \cup \Psi^Y$ will be dense.

Finally, if A and B are both Δ^0_2, we can let $Z_e \not\geq_{\mathrm{T}} A \oplus B$. The argument is the same. ■

4.2 There are minimal pairs for generic reducibility

We saw in the last section that the notion of relative generic computability is "broken enough" that it has no minimal pairs. It turns out that, for generic reducibility, we do have minimal pairs.

Theorem 4.5 ([5], see also [6]). There exists a minimal pair for nonuniform generic reducibility. More explicitly, there exists two sets A_0 and A_1 which are not generically computable, but if B is nonuniformly generically reducible to both A_0 and A_1, then B is generically computable.

We fix a computable enumeration $\{W_e\}_{e\in\mathbb{N}}$ of the enumeration operators.

Proof. We will construct four Δ^0_2 objects: the sets A_0 and A_1, and the generic descriptions f_0 and f_1 of A_0 and A_1, respectively. If W_{e_1} and W_{e_2} are two enumeration operators, the functions f_0 and f_1 will try to diagonalize against $W^{f_0}_{e_0}$ and $W^{f_1}_{e_1}$ being generic descriptions of the same set; or, failing that, making sure that this common set is generically computable in the first place.

For an enumeration operator W_e, we denote by $W^{f_i}_e[s]$ the set of all k such that (F, k) is enumerated at the sth stage of computation of W_e, for some F contained in the sth stage of f_i.

The functions f_i will be 1 wherever defined; this will simplify the argument. First, we define $f_i(i) = 1$ and leave $f_i(1-i)$ undefined for $i = 0, 1$.

Claim 4.6. Under these conditions, if there are indices e_0, e_1 such that $W^{f_0}_{e_0}$ and $W^{f_1}_{e_1}$ are generic descriptions of the same set B, then there exists an index e such that $W^{f_0}_e$ and $W^{f_1}_e$ are also generic descriptions of the set B.

This allows us to consider only one enumeration operator each time, simplifying the notation.

Proof. Essentially, let W^f behave like W_{e_0} if $f(0) = 1$, behave like W_{e_1} if $f(1) = 1$, and not do anything otherwise.

Formally, let

$$\begin{aligned} W = &\Big\{(F \cup \{(0,1)\}, k) \mid (0,1) \notin F \wedge (F,k) \in W_{e_0}\Big\} \\ &\cup \Big\{(F \cup \{(1,1)\}, k) \mid (1,1) \notin F \wedge (F,k) \in W_{e_1}\Big\}. \end{aligned}$$

(Recall that, for enumeration reductions, the sets $W^{f_i}_{e_i}$ must be graphs of characteristic functions, so the elements x are pairs of numbers.) Then $W^{f_0} = W^{f_0}_{e_0}$ and $W^{f_1} = W^{f_1}_{e_1}$. ■

Continuing the proof, let R_e be as in Example 2.22. To make sure that A_0 and A_1 are not generically computable, we will satisfy the following set of requirements:

$$\mathcal{P}_{e,i}: \text{ if } \operatorname{dom}\Phi_e \cap R_e \text{ is infinite, then } \Phi_e(n) \neq A_i(n) \text{ for some } n \in R_e$$

and

$$\mathcal{N}_e: \forall x \forall s \Big[\text{ if } x \in W^{f_0}_e[s] \cap W^{f_1}_e[s], \text{ then either } x \in W^{f_0}_e \text{ or } x \in W^{f_1}_e \Big]$$

The requirements $\mathcal{P}_{e,i}$ make sure that A_i is not generically computable; if these requirements are satisfied, then either $\Phi_i(n) \neq A_i(n)$ for some n, so that Φ_e is not a generic description of A_i, or $\operatorname{dom}\Phi_e \cap R_e$ is finite, so that Φ_e is not a generic description of anything because $\operatorname{dom}\Phi_e$ is not dense.

The requirements $\mathcal{N}_e$ are the fallback if we fail to diagonalize against $W_e^{f_i}$. If $W_e^{f_0}$ and $W_e^{f_1}$ both describe the same set B, define $h(x) = 1$ if $x \in W_e^{f_0}[s] \cap W_e^{f_1}[s]$ for some s, and leave h undefined otherwise. The function h is partial computable, and by requirement $\mathcal{N}_e$, if $h(x) = 1$ then either $x \in W_e^{f_0}$ or $x \in W_e^{f_1}$. So $h = W_e^{f_0} \cup W_e^{f_1}$, which shows that h itself is a generic description of B; hence, B is generically computable.

In isolation, satisfying the requirement $\mathcal{P}_{e,i}$ is easy: we just have to compute successive approximations $\Phi_e[s]$ to Φ_e, and if it converges for some $n \in R_e$, we mark $f_i(n)$ as undefined and define $A_i(n) = 1 - \Phi_e(n)$. (Recall that f_i is 1 where it is defined.) As long as at least one such n is not restricted by higher priority requirements, the requirement $\mathcal{P}_{e,i}$ will be satisfied.

The difficult part is not conflicting with the requirements $\mathcal{N}_e$. Intuitively, $\mathcal{N}_e$ says that, if at some stage s we realize that $W_e^{f_0}(x)[s]$ and $W_e^{f_1}(x)[s]$ agree for some x, then we must commit to preserving this computation in at least one of the two sides.

The fact that the functions f_i are 1 wherever defined makes things easier. In order to preserve the computation $W_e^{f_i}(x)[s]$, let u be the use of this computation; that is, $x \in W_e^{f_i}[s]$ if and only if $(F, x) \in W_e[s]$ for some $F \subseteq f_i$; let $u = \max F$. Then restrict the values of $f_i(n)$ for $n \leq u$ from changing. (Note that we only need to preserve the computation in one of the sides.)

But this also allows us to "restore" computation states: if this computation is violated at a further step t, we can restore the value $W_e^{f_i}(x)[s]$ by simply making F a subset of $f_i[t]$ again. There will never be a conflict of values because f_i is 1 wherever defined. The only issue is that this might re-define the value of $f_i(n)$ for some n, injuring some $\mathcal{P}_{e,i}$.

So, in order to satisfy all requirements, we let the $\mathcal{P}_{e,i}$ issue restraints, rather than the $\mathcal{N}_e$. Specifically, each $\mathcal{P}_{e,i}$ will try to choose some $n \in R_e$ to serve as a witness to $A_i \neq \Phi_e$, and it will make $f_i(n)$ undefined in the process. If $\mathcal{P}_{e,i}$ is allowed to make $f_i(n)$ undefined (that is, it does not violate any restraints), it issues the restraint that no lower-priority requirement may further undefine any $f_{1-i}(k)$ for $k < s$. This guarantees that all computations $W_e^{f_{1-i}}(k)[s]$ for $k < s$ will be preserved, even if the corresponding computations $W_e^{f_i}(k)[s]$ are not.

If $\mathcal{P}_{e,i}$ is injured by a higher-priority requirement, then we define $f_i(n) = 1$ again, which restores the computations as outlined above.

Now, if $\mathcal{P}_{e,i}$ is impeded to act, then we must try to satisfy it by marking $f_i(n)$ as undefined for some larger n. But this eventually will be the case, as if u is the largest use ever preserved by any higher-priority requirement, then $\mathcal{P}_{e,i}$ just needs to undefine some $n > u$.

Finally, neither A_0 nor A_1 is generically computable, and if B is nonuniformly generically reducible to both A_0 and A_1, then there are indices e_0 and e_1 such that $W_{e_0}^{f_0}$ and $W_{e_1}^{f_1}$ are generic descriptions of B, so by the claim there is a single index e such that $W_e^{f_0}$ and $W_e^{f_1}$ are generic descriptions of B, and thus by the requirement $\mathcal{N}_e$, the set B is generically computable. ■

5 There are only a few minimal pairs for generic reducibility

In the proof of Theorem 4.4, we constructed two total Turing functionals Φ and Ψ, and two collection of Δ^0_2 sets X_e and Y_e such that, if $X \neq X_e$ for all e and $Y \neq Y_e$ for all e, then $\Phi^X \cup \Psi^Y$ is a dense set without dense c.e. subsets. In this section we will be plugging in generic descriptions in Φ and Ψ, to provide a measure-theoretic quantification of Hirschfeldt's Theorem 4.5.

Given a partial function $f : \mathbb{N} \to \{0,1\}$, write $X \succcurlyeq f$ if f can be extended to the characteristic function of X (that is, $f(x) = 0$ implies $x \notin X$ and $f(x) = 1$ implies $x \in X$). For any Turing functional Φ define $\mathcal{W}_\Phi$ by $\mathcal{W}^f_\Phi(n) = 1$ if $\Phi^X(n) = 1$ for all $X \succcurlyeq f$, and leave $\mathcal{W}^f_\Phi(n)$ undefined otherwise. For example, if f is the characteristic function of X, then $\mathcal{W}^f_\Phi$ is just Φ^X but with the zeros replaced with "undefined".

We want to use $\mathcal{W}_\Phi$ and $\mathcal{W}_\Psi$ as enumeration operators, where Φ and Ψ are the Turing functionals defined in the proofs of Theorems 4.1 and 4.4.

Proposition 5.1. *If Φ is a Turing functional, then $\mathcal{W}_\Phi$ is an enumeration operator.*

Proof. This follows by compactness. Intuitively, to compute $\mathcal{W}^f_\Phi(n)$, we verify all strings σ which agree with the partial function f (that is, if $f(s)\downarrow$ and $s < |\sigma|$ then $f(s) = \sigma(s)$) whether $\Phi^\sigma(n) = 1$. If $\Phi^X(n) = 1$ for all $X \succcurlyeq f$, then for some length t, all strings σ agreeing with f with length t will satisfy $\Phi^\sigma(n)\downarrow = 1$, so we enumerate $\mathcal{W}^f_\Phi(n) = 1$ at this moment. (If no such t exists, then for each t there exists a string σ of length t agreeing with f for which $\neg(\Phi^\sigma(n)\downarrow = 1)$. Hence by the Weak König's Lemma [12, Theorem 8.3.1] there exists some $X \succcurlyeq f$ for which $\neg(\Phi^X(n)\downarrow = 1)$, so we are correct in not enumerating anything.)

Formally, to enumerate a pair (F, k) into $\mathcal{W}_\Phi$, we first must have $k = 1$, and the finite set F be the graph of a partial $\{0,1\}$-valued function h. (Note that h has finite domain.) Then compute $\Phi^\sigma(n)[|\sigma|]$ for all strings σ which agree with h. If for some length t, all such σ of length t satisfy $\Phi^\sigma(n)[|\sigma|]\downarrow = 1$ then enumerate (F, k). This finishes the construction.

By construction, if $\mathcal{W}^f_\Phi(n)\downarrow = 1$, then $(\operatorname{graph} h, 1) \in \mathcal{W}^f_\Phi$ for some finite $\operatorname{graph} h \subseteq \operatorname{graph} f$, so we have $\Phi^X(n)\downarrow = 1$ for all $X \succcurlyeq h$, and hence $\Phi^X(n)\downarrow = 1$ for all $X \succcurlyeq f$. Conversely, if $\Phi^X(n)\downarrow = 1$ for all $X \succcurlyeq$, by compactness [12, Theorem 8.3.1] there exists a t for which these computations of $\Phi^X(n)$ use only the first t bits of X, and thus the construction above enumerates $(F, 1)$ into $\mathcal{W}_\Phi$ where F is the graph of $f \upharpoonright t$. ■

In the proof of Theorem 4.4, in order to diagonalize against dense c.e. sets, the sets X_e and Y_e were "sacrificed" in the sense that Φ^{X_e} is not dense, but Φ^X is if $X \neq X_e$ for all e. We have a similar result here.

Proposition 5.2. *If $f \neq X_e$ for all e (i.e. for all e there exists some n where $f(n) \downarrow\neq X_e(n)$), then $\mathcal{W}^f_\Phi$ has density 1.*

This implies that $\mathcal{W}_\Phi^f$ is a generic description of all sets containing $\mathcal{W}_\Phi^f$.

Proof. Consider strategy e. If T_e is finite, then this strategy places only finitely many gaps, so assume that T_e is infinite and let n satisfy $f(n) \neq X_e(n)$.

At some stage t, the first $n+1$ bits of X_e will converge (thinking of $(X_{e,s}, Y_{e,s})$ as being the string pair marked by the eth strategy on stage s). Hence beyond stage t, strategy e will only place gaps in Φ^X for $X \not\succcurlyeq f$. Therefore, assuming by induction on e that, on stage t, all strategies $e' < e$ also stopped placing gaps on Φ^X for $X \not\succcurlyeq f$, this means that for any $s > t$, the density of Φ^f in P_s is at least $1 - 2^{-e}$.

By induction, $\mathcal{W}_\Phi^f$ has density 1. ■

Call two sets A and X coarsely similar if the symmetric difference $A \triangle X$ has density zero. If A and X are not coarsely similar, then no generic description of A is also a generic description of X.

Proposition 5.3. *Let A be a set which is not coarsely similar to any X_e and B a set which is not coarsely similar to any Y_e. Then (A, B) do not form a minimal pair for the uniform generic degrees.*

Proof. Let $C = \Phi^A \cup \Psi^B$. The set C has density 1, and by the proof of Theorem 4.4 it contains no density-1 c.e. subset. Thus, C is not generically computable.

However, for any partial $f \preccurlyeq A$ with dense domain, we have $\mathcal{W}_\Phi^f \subseteq C$, and since A is not coarsely similar to any X_e we know that $f \neq X_e$ for all e. By the previous proposition $\mathcal{W}_\Phi^f$ is dense, being thus a generic description of C.

This means that C is generically reducible to A, and analogously C is generically reducible to B. ■

Two sets which are not generically equivalent may still have a common generic description; for example, if C is any density-1 non-generically computable set then C and $\mathbb{N}$ can both be generically described by the function which is 1 in C and undefined otherwise. Hence not being generically equivalent does not imply not being coarsely similar, and thus the above result cannot be used to show that in any minimal pair for the generic degrees at least one side contains a Δ_2^0 set, for example.

But using randomness we can get a measure-theoretic sense of how rare minimal pairs are.

Theorem 5.4. If A and B are both 2-random, then (A, B) does not form a minimal pair for the generic degrees.

This means that the collection of pairs (A, B) which form a minimal pair for the generic degrees has measure zero. This contrasts with the situation for Turing degrees (Theorem 3.4).

Proof. If A is 2-random, then it is 1-random relative to $\emptyset'$, so it is 1-random relative to X_e. By the relativized form of Proposition 2.5, the set A is Church-stochastic with respect to X_e. This means that $A \triangle X_e$ has positive upper density, otherwise A would be coarsely computable relative to X_e, contradicting Proposition 2.24. Similarly, B is not coarsely similar to any Y_e.

Therefore, by Proposition 5.3, the pair (A, B) is not a minimal pair for the generic degrees. ■

6 Functions and Sets

A common theme in both computability and complexity theory is to only talk about yes/no questions, rather than "function questions". For example, the Satisfiability Problem SAT in complexity theory asks whether a Boolean formula has a satisfying assignment or not, whereas in real-world application we would be more interested in finding such an assignment. The reasoning is that this makes the theory more elegant and comes at no cost to generality. For example, we can find a satisfying assignment to a formula with n variables in polynomial time by performing $O(n)$ queries to an oracle for SAT, so the set problem and the function problem are in the same polynomial class of complexity.

Surprisingly, for the four asymptotic notions of computability studied in this paper, it is still an open problem whether functions are equivalent to sets. In this section we take a small step towards answering this question by showing that a certain class of enumeration operators is unable to show that every uniform degree contains a set.

Formally, we define coarse, dense, generic, and effective generic reducibilities and degrees for functions in the same way we define for sets, *mutatis mutandis.* The question can then be stated as follows.

Open Problem 6.1. For each of the eight reducibilities (the uniform and non-uniform versions of dense reducibility, generic reducibility, coarse reducibility, and effective dense reducibility), defined for functions, is it true that every degree contains the indicator function of a set?

Every nonuniform degree is a union of uniform degrees, so if the answer to this question is negative, it should be easier to prove so for uniform reducibility. We may simplify the question further, and ask whether there is a single enumeration operator W such that, for all functions f, the set W^f is the indicator function of a set and f and W^f have the same uniform generic degree. (In a sense, this is a stronger uniformity condition.)

We will show that the the answer is negative for the following restricted class of operators.

Definition 6.2. For the purposes of this section, define a *simple encoding* to be a function $E : \mathbb{N} \to 2^{\mathbb{N}}$ such that $E(x)$ and $E(y)$ are disjoint if $x \neq y$. For any partial function $f : \mathbb{N} \to \mathbb{N}$, define E_f by

$$E_f = \bigcup_{n \in \operatorname{dom} f} E(\langle n, f(n) \rangle).$$

For example, if we let $E(x) = \{x\}$, then E_f is the graph of f. The functions $\mathcal{R}$ and $\tilde{\mathcal{R}}$ can also be thought of simple encodings; for example, if we set $E(\langle n, 1\rangle) = R_n$ and $E(x) = \emptyset$ otherwise, then $E_{\mathbb{1}_A} = \mathcal{R}(A)$ for all sets A.

We have the following.

Theorem 6.3. For each simple encoding E there exists a function f such that f and the indicator function of E_f are not in the same nonuniform coarse, generic, dense, or effectively dense degrees.

Therefore, simple encodings cannot transform a function into a set whilst preserving its degree.

Proof. We will analyze 3 separate cases, according to how the densities of $E(x)$ behaves asymptotically.

Case 1: There are infinitely many n for which $\overline{\rho}(E(\langle n, k\rangle)) > 0$ for some k.

Let $n_0 < n_1 < \cdots$ be an infinite sequence of these numbers n, with corresponding witnesses k_i. We may assume that the set $\mathcal{N} = \{n_0, n_1, \dots\}$ has density 0, as we can replace $\mathcal{N}$ with a sparse subset. For each $\alpha : \mathbb{N} \to \{0, 1\}$ define $f_\alpha : \mathbb{N} \to \mathbb{N}$ by setting $f_\alpha(n_i) = k_i + \alpha(i)$ for each n_i and $f_\alpha(m) = 0$ if $m \notin \mathcal{N}$. We will show that f_α and E_{f_α} are not equivalent for some α.

If $\alpha \neq \beta$, say $\alpha(i) \neq \beta(i)$, then $E_{f_\alpha} \triangle E_{f_\beta}$ contains the set $E(\langle n_i, k_i\rangle)$, which has positive upper density, and thus $E_{f_\alpha} \triangle E_{f_\beta}$ itself has positive upper density. This means that, for each Turing machine e, the function $\Phi_e^{\mathcal{N}}$ can be a dense description of at most one of the sets E_{f_α}.

There are uncountably many such E_{f_α}, but only countably many $\mathcal{N}$-computable dense descriptions, so for at least some α the set E_{f_α} is not densely computable relative to $\mathcal{N}$. However, because $\mathcal{N}$ has density 0, the function f_α itself is effectively densely computable relative to $\mathcal{N}$. This means that even relative to $\mathcal{N}$, the nonuniform coarse, dense, effectively dense, and generic degrees of the function f_α do not contain the indicator function of the set E_{f_α}.

Case 2: There exists some $\varepsilon > 0$ such that, for infinitely many n, there exists some k and j where $\rho_j(E(\langle n, k\rangle)) > \varepsilon$.[4]

Again, let $n_0 < n_1 < \cdots$ be an infinite sequence of such n, with k_i and j_i being the corresponding witnesses, and assume that $\mathcal{N} = \{n_0, n_1, \dots\}$ is sparse. For $\alpha : \mathbb{N} \to \{0, 1\}$ define f_α as before. Now, suppose that α and β differ at infinitely many places; we claim that $E_{f_\alpha} \triangle E_{f_\beta}$ has positive upper density.

Indeed, the set $E_{f_\alpha} \triangle E_{f_\beta}$ contains infinitely many sets $E(\langle n_i, k_i\rangle)$, each satisfying $\rho_{j_i}(E(\langle n_i, k_i\rangle)) > \varepsilon$ for some integer k_i. It is easy to see that if $\rho_{j_i}(E(\langle n_i, k_i\rangle)) > 0$ then the set $E(\langle n_i, k_i\rangle)$ must contain some element smaller than j_i. Since all $E(\langle n_i, k_i\rangle)$ are disjoint, this means that each j_i may only be repeated finitely many times (that is, for each j, the set $\{i \mid j_i = j\}$ is finite). Hence there are infinitely many distinct j_i such that $\rho_{j_i} E(\langle n_i, k_i\rangle) > \varepsilon$ in any infinite subcollection of the j_i. In turn, this means that $\rho_{j_i}(E_{f_\alpha} \triangle E_{f_\beta}) > \varepsilon$ for infinitely many distinct j_i, which shows that the set $E_{f_\alpha} \triangle E_{f_\beta}$ has positive upper density.

[4]There is overlap between Case 1 and Case 2, but this is neither harmful nor relevant.

Since there are uncountably many α which differ in infinitely many places, there are uncountably many E_{f_α} whose pairwise symmetric differences all have positive upper density. Therefore, the same conclusion as in Case 1 applies.

Case 3: Neither Case 1 nor Case 2 applies.

Because we are not in Case 1, there exists some N such that $\overline{\rho}(E(\langle n,k\rangle)) = 0$ for all k and all $n \geq N$.

If $n \geq N$, define ε_n by

$$\varepsilon_n = \sup_{k,j\in\mathbb{N}} \rho_j(E(\langle n,k\rangle)).$$

Because we are not in Case 2, we have $\lim_{n\to\infty} \varepsilon_n = 0$.

Construct the sequences $k_{N,i}, k_{N+1,i}, k_{N+2,i}, \ldots$ for $i = 0,1$ as follows. First, let $k_{N,0} \neq k_{N,1}$ be arbitrary. Assume we defined $k_{N,i}, k_{N+1,i}, \ldots, k_{n-1,i}$ for $i = 0,1$, and define

$$A_n = \bigcup_{\substack{N\leq \ell<n\\ i=0,1}} E(\langle \ell, k_{\ell,i}\rangle).$$

Because A_n is a union of finitely many $E(\langle \ell,k\rangle)$ for $\ell \geq N$, all of which have density 0, we have that $\rho(A_n) = 0$. Therefore, there is some positive M_n such that $\rho_j(A_n) < \varepsilon_n$ for all $j > M_n$. Choose $k_{n,0}$ and $k_{n,1}$ to be any two distinct integers such that neither $E(\langle n, k_{n,0}\rangle)$ nor $E(\langle n, k_{n,1}\rangle)$ contains elements smaller than M_n or N; such $k_{n,i}$ exist because the $E(\langle n,k\rangle)$ are all disjoint. This implies that, if $j > M_n$, then

$$\rho_j(A_{n+1}) = \rho_j(A_n) + \rho_j(E(\langle n, k_{n,0}\rangle)) + \rho_j(E(\langle n, k_{n,1}\rangle)) < 3\varepsilon_n.$$

Define $A = \bigcup_n A_n$. We may assume that the numbers M_n are increasing. If $M_n \leq j < M_{n+1}$, then

$$\rho_j(A) = \rho_j(A_{n+1}) \leq 3\varepsilon_n.$$

This shows that $\rho(A) = 0$.

Now, finally, given $\alpha : \mathbb{N} \to \mathbb{N}$, define $f_\alpha(n) = 0$ if $n < N$ and $f_\alpha(n) = k_{n,\alpha(n-N)}$ otherwise. For any two α, β, we have $E_{f_\alpha} \triangle E_{f_\beta} \subseteq A$, so all of the E_{f_α} are effectively densely computable relative to A. If α and β disagree on a set with positive upper density, then f_α and f_β cannot be A-densely computed by the same Turing machine. Since there is an uncountable family of functions α which pairwisely disagree on a set with positive upper density, there are functions α such that f_α is not A-densely computable, which in turn mean that there are functions α whose the nonuniform coarse, dense, effectively dense, and generic degrees do not contain the indicator function of the set E_{f_α}. ■

7 Open Problems

As mention in Section 6, it is not known whether sets are equivalent to functions. We restate this question here, for completeness.

Open Problem 6.1. For each of the eight reducibilities (the uniform and nonuniform versions of dense reducibility, generic reducibility, coarse reducibility, and effective dense reducibility), defined for functions, is it true that every degree contains the indicator function of a set?

The statement of Theorem 5.4 raises the following question.

Open Problem 7.1. Theorem 5.4 requires the sets to be 2-random. Can this condition be improved to 1-random?

In Igusa's proof of Theorem 4.1, we could have replaced "undefined" with $\square$ throughout, resulting in the theorem that there are no minimal pairs for relative effectively dense computability. Similarly, the same modifications applied to Theorem 5.4 yields the result that if A and B are 2-random then A and B do not form a minimal pair for effectively dense reducibility. However, the existence of minimal pairs is still open.

Open Problem 7.2. Do there exists minimal pairs for the effective dense degrees (both uniform and nonuniform)?

In Section 2.4, we provided examples of sets which are densely computable but neither coarsely nor generically computable (Example 2.26), which are coarsely computable but not densely computable (Example 2.22) and vice-versa (Example 2.23), and which are coarsely and generically computable but not effectively densely computable (Example 2.28). These non-implications between these asymptotic notions of computability entails non-implications in the associated reducibilities. For example, if A is the set from Example 2.26 then $A \leq_{\mathrm{ud}} \emptyset$, but $A \not\leq_{\mathrm{nc}} \emptyset$ and $A \not\leq_{\mathrm{ng}} \emptyset$. However, the other directions are still open.

Open Problem 7.3. Does $A \leq_{\mathrm{ned}} B$ implies $A \leq_{\mathrm{nc}} B$ or $A \leq_{\mathrm{ng}} B$? Does $A \leq_{\mathrm{nc}} B$ and $A \leq_{\mathrm{ng}} B$ implies $A \leq_{\mathrm{nd}} B$? What about uniform reducibilities?

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