# Quantum Merlin-Arthur and proofs without relative phase 

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#### Abstract

We study a variant of QMA where quantum proofs have no relative phase (i.e. non-negative amplitudes, up to a global phase). If only completeness is modified, this class is equal to QMA [GKS14]; but if both completeness and soundness are modified, the class (named QMA ${ }^{+}$by Jeronimo and Wu [JW23]) can be much more powerful. We show that QMA ${ }^{+}$with some constant gap is equal to NEXP, yet QMA ${ }^{+}$ with some other constant gap is equal to QMA. One interpretation is that Merlin's ability to "deceive" originates from relative phase at least as much as from entanglement, since $\mathrm{QMA}(2) \subseteq$ NEXP.


## 1 Introduction

The strangeness of quantum states has at least two fundamental sources: entanglement, the source of "spooky action at a distance"; and relative phase, which allows for destructive interference. We use complexity theory to probe these sources of strangeness. Extending the main result of [JW23], we find that QMA ${ }^{+}$(QMA where quantum proofs have no relative phase) is as powerful as NEXP.
$\mathrm{A}_{\mathrm{QMA}}^{c, s}$ protocol is a verification task for a quantum computer (termed "Arthur") when interacting with a dishonest but all-powerful machine (termed "Merlin"). If the statement is true ("completeness"), Merlin sends a quantum state ("proof") that truthfully convinces Arthur. If the statement is false ("soundness"), Merlin will send any quantum state possible to deceive Arthur. A valid protocol distinguishes these cases, succeeding with probability at least $c$ in completeness and at most $s<c$ in soundness. Canonically, QMA is the class of all valid $\mathrm{QMA}_{2 / 3,1 / 3}$ protocols.

One could potentially reduce the power of QMA by restricting Merlin's proof in completeness. Surprisingly, many restrictions of this type do not reduce the power of the class. For example, this is true even if the quantum state is a subset state (with no relative phase nor relative non-zero amplitude) [GKS14]. The reason behind this is promise gap amplification: there exist techniques to increase the gap $c-s$ to $1-2^{-p(n)}$ for any polynomial $p(n)$. As a result, a subset state with polynomially small overlap with the best completeness proof succeeds. This argument generalizes to any set of states that form an $\epsilon_{n}$-covering of all $n$-qubit quantum states, where $\epsilon_{n}$ is at least inverse polynomial in $n$.

By contrast, restricting Merlin's proof in soundness seems to increase the power of this complexity class, since this reduces Merlin's ability to "deceive". For example, if Merlin must send a quantum proof without relative phase, Arthur can ask about its sparsity ( $\ell_{1}$ norm). When a state has no relative phase, a low overlap with $|+\rangle^{\otimes n}$ actually implies it is sparse, as opposed to a state with destructively interfering phases (i.e. any other Hadamard basis vector).

One popular variant of QMA restricts Merlin's entanglement over a fixed barrier; it is named QMA(2) (as if there are two unentangled Merlins each sending a quantum proof [KMY03]). This complexity class may seem more powerful than QMA, but despite much study [BT10, ABD ${ }^{+} 08, \mathrm{CD} 10, \mathrm{Per} 12$, HM13, GSS ${ }^{+} 18, \mathrm{SY} 22$ ], little is known except the trivial bounds $\mathrm{QMA} \subseteq$ QMA $(2) \subseteq$ NEXP.

What happens if one restricts both entanglement and relative phase? [JW23] define QMA ${ }_{c, s}^{+}$and $\mathrm{QMA}^{+}(2)_{c, s}$, where quantum proofs are required to have no relative phase (non-negative amplitudes, up to a global

[^0]
## QMA(2) $\longrightarrow$ QMA(2) ${ }^{+}$



Figure 1: Relationship between $\mathrm{QMA}^{+}$and $\mathrm{QMA}(2)$. [JW23] show that for some constants $1>c>s>0, \mathrm{QMA}^{+}(2)_{c, s}=$ NEXP. We show that the same is true for QMA ${ }^{+}$. Restricting relative phase does not restrict entanglement across a fixed barrier: for example, consider the GHZ state $|0\rangle^{\otimes n}+|1\rangle^{\otimes n}$, or more generally states where the Schmidt vectors have no relative phase.
phase) in both cases. ${ }^{1}$ Surprisingly, [JW23] show the existence of constants $1>c>s>0$ such that QMA $^{+}(2)_{c, s}=$ NEXP, crucially including a protocol to estimate the sparsity of a quantum proof. This hints perhaps at a route to prove $\mathrm{QMA}(2)=$ NEXP, since there are other constants $1>c^{\prime}>s^{\prime}>0$ where $\mathrm{QMA}^{+}(2)_{c^{\prime}, s^{\prime}}=\mathrm{QMA}(2) .{ }^{2}$

In this work, we show that restricting relative phase alone gives the power of NEXP; i.e., there exist constants $1>c>s>0$ where QMA $_{c, s}^{+}=$NEXP. Note that assuming EXP $\neq$NEXP, this implies QMA ${ }^{+}$ cannot be amplified, since as before, there are other constants $1>c^{\prime}>s^{\prime}>0$ where QMA $_{c^{\prime}, s^{\prime}}^{+}=$QMA $\subseteq$ EXP. As a result, techniques to prove $\mathrm{QMA}(2)=\mathrm{QMA}^{+}(2)=$ NEXP must crucially use the unentanglement promise inherent in QMA(2). See Figure 1 and Figure 2 for a pictorial description.

### 1.1 Techniques

Our primary technical contribution is to show a QMA ${ }^{+}$protocol for a NEXP-complete problem. This directly extends the work of Jeronimo and Wu [JW23], who show a QMA ${ }^{+}(2)$ protocol for a NEXP-complete problem. As in [JW23], we study constraint satisfaction problems (CSPs) with constant gap. In ( $1, \delta$ )-GAPCSP, either all constraints can be satisfied, or at most a $\delta$ fraction of constraints can be satisfied. These problems are known to be NP-hard or NEXP-hard (depending on the problem size) using the PCP theorem [AS92, $\mathrm{ALM}^{+} 98$, Har04].

Before proving QMA ${ }^{+}$with some constant gap equals NEXP, we prove QMA ${ }_{\text {log }}^{+}$(with some other constant gap) equals NP. This choice (also taken by [JW23]) is pedagogical: it allows us to explain the protocol without worrying about input encoding size, since $\mathrm{QMA}^{+}$has a polynomial amount of space and verifier runtime. Here, we consider $(1, \delta)$-GAPCSP with polynomially many variables and clauses; the quantum proof must certify that there is a satisfying assignment to all clauses.

The QMA $_{\log }^{+}(2)$ protocol of [JW23] crucially relies on an estimate of sparsity ( $\ell_{1}$ norm) of a quantum state without relative phase. The overlap of a $m$-qubit quantum state without relative phase $|\psi\rangle$ with $|+\rangle^{\otimes m}$ is exactly the value $2^{-m / 2} \cdot \||\psi\rangle \|_{1}$. With multiple quantum proofs $\left|\psi_{1}\right\rangle \otimes \ldots\left|\psi_{k}\right\rangle$, one can estimate the sparsity by repeating this "sparsity test" on each $\left|\psi_{i}\right\rangle$, and using a swap test to ensure that all $\left|\psi_{i}\right\rangle$ are approximately equal. Interestingly, no other part of their protocol requires the no relative phase assumption. ${ }^{3}$

In QMA $_{\log }^{+}$, we have a single quantum proof, so we cannot use this test to estimate sparsity. Instead, we design a similar test that directly enforces a rigidity property of the proof. ${ }^{4}$ The required form is $\frac{1}{\sqrt{R}} \sum_{j \in[R]}|j\rangle\left|\vec{v}_{j}\right\rangle$, where the second register is constant-sized. Using a "sparsity test" over the second

[^1]register, Arthur ensures that the second register has one $\vec{v}_{j}$ per $j$; but using the complement of a "sparsity test" over the whole proof, Arthur ensures the overall state maximizes $\ell_{1}$ norm. States of the required form are optimal for this combination of tests. We make use of the no relative phase property in Lemmas 6 and 8 .

Now we can describe our protocol. For each constraint $j \in R$, Arthur asks for the values $\vec{v}_{j}$ associated with the variables involved in constraint $j$. The protocol either enforces rigidity of the quantum proof, or verifies the constraints of the CSP. Note that we need two kinds of constraint checks: the values $\vec{v}_{j}$ must satisfy constraint $j$, and the value of a variable must be consistent across the constraints it participates in. For states with the rigidity property, checking satisfiability is simple: measure in the computational basis and verify the measured constraint $j, \vec{v}_{j}$. States with the rigidity property will succeed with probability equal to the satisfying fraction of the CSP assignment.

Checking consistency is done using a technique called "regularization" from the PCP literature [Din07]; for each constraint $j$, we verify that each variable participating in $j$ has the same value in exactly $d$ other constraints for some constant $d$, in a way that the edges form an expander graph. The expansion property guarantees that cheating on this test is as damaging as cheating on the satisfiability test. Jeronimo and Wu [JW23] use a swap test to implement these new checks, but this requires multiple quantum proofs. We show how to use a Hadamard test (which requires only one quantum proof) to achieve the same result, building on ideas from previous work [BFM22]. Since there exists a $\delta$ such that $(1, \delta)$-GAPCSP is NP-hard, this completes the proof of $N P \subseteq Q M A_{\text {log }}^{+}$with some constant gap.

When scaling up to $\mathrm{QMA}^{+}$, one must be careful of how to succinctly encode the input of a NEXPcomplete problem. The PCP theorem allows us to choose $(1, \delta)$-GAPCSP that is succinct, but we need stronger properties. Following the adjustments taken in [JW23], we choose a PCP system for NEXP that is both doubly explicit and strongly uniform. Doubly explicit means that one can efficiently compute the variables participating in a given constraint and the constraints a given variable participates in; using this, we can implement the consistency checks in polynomial time. Strongly uniform means that the number of constraints a variable participates in is efficiently computable, and one of a fixed number of possibilities; using this, we only need to build a fixed number of expander graphs during regularization. Recent work also shows how to construct exponentially-sized expander graphs in polynomial time [Lub09, Alo21]. Once we are through these input encoding difficulties, our protocol is identical to that for NP.

Fundamentally, the no relative phase property allows Arthur to verify a number of constraints exponential in the number of qubits. Attempts to do this for $\mathrm{QMA}(2)$ gave too small a promise gap [BT10, Per12, GNN12], too many provers [ABD $\left.{ }^{+} 08, \mathrm{CD} 10, \mathrm{CF} 11\right]$, or too much space or time [HM13, NZ23]. Jeronimo and Wu [JW23] show that $\mathrm{QMA}^{+}(2)$ circumvents this difficulty: using no relative phase and unentanglement, Arthur enforces the sparsity of a quantum proof to solve a NEXP-complete problem. At the center of our work is the insight that no relative phase is enough for Arthur to require constant-sized answers to exponentially many questions, solving a NEXP-complete problem with a single polynomial-size quantum proof.

### 1.2 Related work

The complexity class QMA(2) The complexity class QMA(2) is known to have promise gap amplification, and to be equal to $\operatorname{QMA}(k)$ for any $k$ at most polynomial in $n$ [HM13]. It is not obvious how to test for entanglement; even determining whether a polynomially-sized vector is entangled is NP-hard [Gha09]. If there exist efficient approximate "disentanglers" that can create any separable state, then $\mathrm{QMA}=\mathrm{QMA}(2)$; see $\left[\mathrm{ABD}^{+} 08\right]$ for some progress. [GSS $\left.{ }^{+} 18\right]$ describe quantum variants of the polynomial hierarchy and connect their properties to bounds on $\mathrm{QMA}(2)$. It is not even known whether there is a quantum oracle separating QMA and QMA(2) [Aar21].

PCPs and expander graphs Probabilistically checkable proofs (PCPs) show hardness for CSPs with a constant gap [AS92, $\mathrm{ALM}^{+} 98$, Har04]. Dinur [Din07] proves the PCP theorem using a regularization step, which adds new constraints associated with the edges of a regular expander graph. Polynomial-time regularization for NEXP requires an efficient description of exponentially-sized expander graphs. Recent advances in expander graph constructions [Lub09, Alo21] allow for this type of regularization, first used in [JW23].


Figure 2: Plot of QMA ${ }_{c, s}^{+}$for increasing $\frac{c}{s}$. By our main result, there exist constants $c, s$ where $1<\frac{c}{s}<4$ and QMA $_{c, s}^{+}=$NEXP, but by Corollary $30, \mathrm{QMA}_{c, s}^{+}=$QMA when $\frac{c}{s}>4$. Gap amplification of QMA ${ }^{+}$would imply QMA $=$NEXP.

Quantum states and relative phase Up to a global phase, states with non-negative amplitudes are equivalent to states with no relative phase. [JW23] propose the class $\mathrm{QMA}^{+}$and $\mathrm{QMA}^{+}(2)$, and show QMA $^{+}(2)=$ NEXP. Note that by contrast, QMA restricted to states with real amplitudes is equal to QMA [McK13]. Relative phase was recently proposed as a quantum resource [Xu23]. For both QMA and QMA(2), restricting Merlin in completeness to send a subset state does not change the power of the complexity class (i.e., QMA $=$ SQMA and $\mathrm{QMA}(2)=\operatorname{SQMA}(2))$. [GKS14] also shows why their proof strategy fails if Merlin is restricted in both completeness and soundness.

Rigidity and games Rigidity was first formally introduced in the context of non-local games [MY04], and have been used to prove several complexity class equalities. For example, the CHSH game [CHSH69] tests for a maximally entangled state on two qubits [Tsi93], and was used to prove QMIP $=$ MIP* [RUV13]. The Mermin-Peres magic square game tests for two copies of a maximally entangled quantum state, and was used to prove MIP* $=$ RE [JNV $\left.{ }^{+} 21\right]$. Rigidity is known to exist in broader contexts, including some (but not all) linear constraint games [CMMN20] and monogamy-of-entanglement games [BC23].

## 2 Our setup

We restate the definition of $\mathrm{QMA}^{+}(k)$ from [JW23]. When the proof length is not specified, it is allowed to be at most any polynomial in input size. We follow the conventions $\mathrm{QMA}^{+}:=\mathrm{QMA}^{+}(1), \mathrm{QMA}^{+}(k):=$ $\bigcup_{c-s=\Omega(1)} \mathrm{QMA}^{+}(k)_{c, s}$, and $\mathrm{QMA}_{\log }^{+}:=\mathrm{QMA}^{+}$with proof length at most $O(\log n)$.

Definition $1\left(\mathrm{QMA}_{\ell}^{+}(k)_{c, s}\right)$. Let $k: \mathbb{N} \rightarrow \mathbb{N}$ and $s, c, \ell: \mathbb{N} \rightarrow \mathbb{R}^{+}$be polynomial time computable functions. A promise problem $\mathcal{L}_{y \text { yes }}, \mathcal{L}_{n o} \subseteq\{0,1\}^{*}$ is in $\mathrm{QMA}_{\ell}^{+}(k)_{c, s}$ if there exists a BQP verifier $V$ such that for every $n \in \mathbb{N}$ and every $x \in\{0,1\}^{n}$ :

- Completeness: if $x \in \mathcal{L}_{y e s}$, then there exist unentangled states $\left|\psi_{1}\right\rangle, \ldots,\left|\psi_{k(n)}\right\rangle$, each on at most $\ell(n)$ qubits and with real non-negative amplitudes, s.t.

$$
\operatorname{Pr}\left[V\left(x,\left|\psi_{1}\right\rangle \otimes \ldots \otimes\left|\psi_{k(n)}\right\rangle\right) \text { accepts }\right] \geq c(n)
$$

- Soundness: If $x \in \mathcal{L}_{\text {no }}$, then for every set of unentangled states $\left|\psi_{1}\right\rangle, \ldots,\left|\psi_{k(n)}\right\rangle$, each on at most $\ell(n)$ qubits and with real non-negative amplitudes, we have

$$
\operatorname{Pr}\left[V\left(x,\left|\psi_{1}\right\rangle \otimes \ldots \otimes\left|\psi_{k(n)}\right\rangle\right) \text { accepts }\right] \leq s(n)
$$

We make a few remarks on this complexity class, with extended discussion in Section 5. First, we stress that the restriction to quantum proofs with non-negative amplitudes is promise-symmetric, i.e. both in completeness and in soundness. This is unlike, for example, the class SQMA [GKS14]. Although the restriction to subset states is stronger than non-negative amplitudes, ${ }^{5}$ its use only in completeness allows for

[^2]SQMA = QMA. In fact, our work implies that QMA with a promise-symmetric subset state restriction also interpolates from QMA to NEXP, depending on the size of the promise gap.

We also explain why the promise gap of $\mathrm{QMA}^{+}$cannot obviously be amplified. The first strategy one might try is parallel repetition: an honest Merlin sends multiple copies of the original proof and Arthur verifies each copy of the original proof. For QMA, entangling the copies in soundness does not help Merlin, since Arthur's protocol is sound for all quantum states. But perhaps unintuitively, it can help for QMA $^{+}$. (See Fact 28 for a simple example.) This is because partial measurement can destroy the restriction on the quantum proof! For example, Arthur's first measurement may introduce relative phase in the rest of the proof. This fact also obstructs more clever amplification strategies for QMA such as the proof-length preserving variant [MW05].

Furthermore, it is not clear how to upper-bound QMA $^{+}$beyond the trivial NEXP. ${ }^{6}$ One technique to upper-bound QMA is to find the optimal proof using a semidefinite program (or a general convex program). This shows that QMA $\subseteq$ PSPACE (or QMA $\subseteq$ EXP with a convex program). But these arguments do not immediately transfer to $\mathrm{QMA}^{+}$. Convex optimization over states with non-negative amplitudes is equivalent to optimizing over the copositive cone [Bur11]. Even the weak membership problem over the copositive cone (deciding if the optimal vector is close to a non-negative vector) is NP-hard in polynomially-sized vector spaces; recall that quantum states are in exponentially-sized vector spaces. These are the same reasons that prevent straightforward upper bounds for QMA(2) [Gha09].

## 3 QMA $_{\log }^{+}$Protocol for NP

We first define the problem we consider:
Definition 2 (CSP system). $A(N, R, q, \Sigma)$-CSP system $\mathcal{C}$ on $N$ variables with values in $\Sigma$ consists of a set (possibly a multi-set) of $R$ constraints $\left\{\mathcal{C}_{1}, \ldots, \mathcal{C}_{R}\right\}$ where the arity of each constraint is exactly $q$.

Definition 3 (Value of CSP). The value of $a(N, R, q, \Sigma)-C S P$ system $\mathcal{C}$ is the maximum fraction of satisfiable constraints over all possible assignments $\sigma:[N] \rightarrow \Sigma$. The value of $\mathcal{C}$ is denoted $\operatorname{val}(\mathcal{C})$.
Definition 4 (GAPCSP). The $(1, \delta)$-GapCSP problem inputs a CSP system $\mathcal{C}$. The task is to distinguish whether $\mathcal{C}$ is such that (in completeness) $\operatorname{val}(\mathcal{C})=1$ or (in soundness) $\operatorname{val}(\mathcal{C}) \leq \delta$.

Fix the input size $n$, and consider $(N, R, q, \Sigma)$-CSP systems where $N$ and $R$ are polynomials in $n$. Deciding whether or not these systems are satisfiable is NP-hard. In fact, there exists $\delta<1$ such that deciding $(1, \delta)$-GapCSP on these CSP systems is NP-hard.
Theorem 5 ([Din07]). There exist constants $q>1$ and $|\Sigma|$ such that (1,1/2)-GAPCSP is NP-hard.
Our goal in this section is to construct a protocol for $(1, \delta)$-GAPCSP given any $(N, R, q, \Sigma)$-CSP system $\mathcal{C}$ where $N, R=\operatorname{poly}(n)$ and $q,|\Sigma|=O(1)$. Let $\kappa:=|\Sigma|^{q}$. We first outline the protocol. Arthur asks for a quantum state from $\mathbb{C}^{R} \otimes \mathbb{C}^{\kappa}$; we call the first register the constraint register and the second register the color register. A quantum proof has the following form:

$$
|\psi\rangle:=\sum_{j \in[R], x \in \Sigma^{q}} a_{j, x}|j\rangle|x\rangle
$$

For completeness, consider the satisfying assignment of variables (in [ $N$ ]) to values (in $\Sigma$ ). Merlin sends the quantum proof $\frac{1}{\sqrt{R}} \sum_{j \in[R]}|j\rangle\left|\vec{v}_{j}\right\rangle$, where each $\vec{v}_{j}$ is the (ordered) list of values associated with the variables participating in $\mathcal{C}_{j}$.

Arthur then applies one of two kinds of tests:

1. Rigidity tests: These ensure that the quantum proof is of the form $\frac{1}{\sqrt{R}} \sum_{j \in[R]}|j\rangle\left|\vec{v}_{j}\right\rangle$.
2. Constraint tests: These verify that values in the quantum proof satisfy constraints of the CSP system.

Below, we separately describe the rigidity tests and constraint tests. In each, we analyze the success probability in completeness and prove lemmas to study soundness. We then combine the technical statements to prove the result.

[^3]
### 3.1 Rigidity tests

Arthur enforces rigidity of a quantum proof using two tests. The first test is the Density test, which maximizes $\ell_{1}$ norm. Here, we measure the state in the Hadamard basis and accept if the outcome is $|+\rangle .{ }^{7}$ Given $|\psi\rangle$, the success probability of this test is

$$
D(|\psi\rangle)=|\langle+\mid \psi\rangle|^{2}=\frac{1}{\kappa R}\left|\sum_{j \in[R], x \in \Sigma^{q}} a_{j, x}\right|^{2}=\frac{1}{\kappa R}\left(\||\psi\rangle \|_{1}\right)^{2}
$$

Recall that if $|\psi\rangle$ is a subset state according to subset $S$, its sparsity $\||\psi\rangle \|_{1}$ is exactly $\sqrt{|S|}$. In completeness, the quantum proof is a subset state with $R$ elements, so this test passes with probability $\frac{1}{\kappa}$.

The second test is the Validity test, which minimizes $\ell_{1}$ norm only on the second register. Here, we measure the color register in the Hadamard basis, and reject if the outcome is $|+\rangle$. Given $|\psi\rangle$, the success probability of this test is

$$
V(|\psi\rangle)=1-\langle+| \operatorname{Tr}_{R}(|\psi\rangle\langle\psi|)|+\rangle=1-\frac{1}{\kappa} \sum_{j \in[R]}\left|\sum_{x \in \Sigma^{q}} a_{j, x}\right|^{2}
$$

where $\operatorname{Tr}_{R}$ is partial trace over the constraint register. In completeness, recall that the proof has the form $\frac{1}{\sqrt{R}} \sum_{j \in[R]}|j\rangle\left|\vec{v}_{j}\right\rangle$, so the success probability is

$$
1-\langle+|\left(\frac{1}{R} \sum_{j \in[R]}\left|\vec{v}_{j}\right\rangle\left\langle\vec{v}_{j}\right|\right)|+\rangle=1-\frac{1}{\kappa}
$$

In fact, no quantum state without relative phase can pass the Validity test with a higher probability:
Lemma 6. Suppose $|\psi\rangle$ has no relative phase. Then $V(|\psi\rangle) \leq 1-\frac{1}{\kappa}$.
Proof. The success probability $V(|\psi\rangle)$ is

$$
1-\frac{1}{\kappa} \sum_{j \in[R]}\left(\sum_{x \in \Sigma^{q}} a_{j, x}\right)^{2}=1-\frac{1}{\kappa}\left(\sum_{j \in[R]} \sum_{x, y \in \Sigma^{q}} a_{j, x} a_{j, y}\right) \quad=1-\frac{1}{\kappa}\left(1+\sum_{j \in[R]} \sum_{x, y \in \Sigma^{q} ; x \neq y} a_{j, x} a_{j, y}\right) \leq 1-\frac{1}{\kappa},
$$

where the second equality follows from $\sum_{j, x} a_{j, x}^{2}=1$ and the inequality holds since $a_{j, x} \geq 0$.
It turns out that is impossible to score high on both the Validity test and the Density test. We use this to enforce the rigidity property of $|\psi\rangle$.

Lemma 7. $D(|\psi\rangle)+V(|\psi\rangle) \leq 1$.
Proof. By Cauchy-Schwarz,

$$
D(|\psi\rangle)=\frac{1}{\kappa R}\left|\sum_{j \in[R]} \sum_{x \in \Sigma^{q}} a_{j, x}\right|^{2} \leq \frac{1}{\kappa} \sum_{j \in[R]}\left|\sum_{x \in \Sigma^{q}} a_{j, x}\right|^{2}=1-V(|\psi\rangle) .
$$

Why does this help with rigidity? Suppose Arthur inputs a quantum proof (without relative phase) $|\psi\rangle$ and runs Density test with probability $p_{1}$ and Validity test with probability $p_{2}$. Suppose also that $p_{2}>p_{1}$. Then the expected success probability is $p_{1} D(|\psi\rangle)+p_{2} V(|\psi\rangle) \leq p_{1}+\left(p_{2}-p_{1}\right) V(|\psi\rangle) \leq p_{1}+\left(p_{2}-p_{1}\right)\left(1-\frac{1}{\kappa}\right)$. Note that this upper bound is achieved in completeness, and for any state of the form $\frac{1}{\sqrt{R}} \sum_{j \in R}|j\rangle\left|\vec{v}_{j}\right\rangle$. We show that quantum proofs must have this form to reach the upper bound.

One requirement to get close to the upper bound is near-optimal success probability on Validity test. We prove that any quantum proof that has this property must be close to a state that assigns one color to each constraint.

[^4]Lemma 8. Given $|\psi\rangle=\sum_{j, x} a_{j, x}|j\rangle|x\rangle$ with no relative phase (i.e. $a_{j, x} \geq 0$ ), let

$$
\gamma:=\max _{\nu:[R] \rightarrow \Sigma^{q}} \sum_{j \in[R]} a_{j, \nu(j)}^{2}
$$

be associated with maximizing function $\sigma$, and let $|\phi\rangle:=\frac{1}{\sqrt{\gamma}} \sum_{j} a_{j, \sigma(j)}|j\rangle|\sigma(j)\rangle$. Fix any $d \geq 0$. If $V(|\psi\rangle)=$ $1-\frac{1+d}{\kappa}$, then $|\langle\psi \mid \phi\rangle|^{2} \geq 1-d$.

Proof. Note that for all $x \in \Sigma^{q}, a_{j, \sigma(j)} \geq a_{j, x}$; otherwise, $\sigma$ is not maximizing. Using the proof of Lemma 6,

$$
d=\sum_{j \in[R]} \sum_{x, y \in \Sigma^{q} ; x \neq y} a_{j, x} a_{j, y} \geq \sum_{j \in[R]} \sum_{y \in \Sigma^{q} ; y \neq \sigma(j)} a_{j, \sigma(j)} a_{j, y} \geq \sum_{j \in[R]} \sum_{y \in \Sigma^{q} ; y \neq \sigma(j)} a_{j, y}^{2}
$$

So then,

$$
\gamma=\sum_{j \in[R]} a_{j, \sigma(j)}^{2}=\left(\sum_{j \in[R]} \sum_{x \in \Sigma^{q}} a_{j, x}^{2}\right)-\left(\sum_{j \in[R]} \sum_{x \in \Sigma^{q} ; x \neq \sigma(j)} a_{j, x}^{2}\right) \geq 1-d .
$$

So $|\langle\psi \mid \phi\rangle|^{2}=\left(\frac{1}{\sqrt{\gamma}} \sum_{j} a_{j, \sigma(j)}^{2}\right)^{2}=\gamma \geq 1-d$.
Another requirement to get close to the upper bound is near-optimal success probability on Density test, up to Lemma 7. Consider any quantum proof that passes Validity test with probability close to $1-\frac{1}{\kappa}$ and Density test with probability close to $\frac{1}{\kappa}$; we show it must be close to a state of the form $\frac{1}{\sqrt{R}} \sum_{j \in R}|j\rangle\left|\vec{v}_{j}\right\rangle$. Now we can prove the soundness of the rigidity test by relying on the following fact:

Fact 9. Let $0 \leq \Pi \leq \mathbb{I}$ be a positive semi-definite matrix, and let $\left|\psi_{1}\right\rangle$ and $\left|\psi_{2}\right\rangle$ be quantum states such that $\left|\left\langle\psi_{1} \mid \psi_{2}\right\rangle\right|^{2} \geq 1-d$. Then $\left.\left|\left\langle\psi_{1}\right| \Pi\right| \psi_{1}\right\rangle-\left\langle\psi_{2}\right| \Pi\left|\psi_{2}\right\rangle \mid \leq \sqrt{d}$.
Proof. The quantity $\left.\left|\left\langle\psi_{1}\right| \Pi\right| \psi_{1}\right\rangle-\left\langle\psi_{2}\right| \Pi\left|\psi_{2}\right\rangle\left|=\left|\operatorname{Tr}\left(\Pi\left(\left|\psi_{1}\right\rangle\left\langle\psi_{1}\right|-\left|\psi_{2}\right\rangle\left\langle\psi_{2}\right|\right)\right)\right|\right.$ is upper-bounded by the trace distance of $\left|\psi_{1}\right\rangle\left\langle\psi_{1}\right|$ and $\left|\psi_{2}\right\rangle\left\langle\psi_{2}\right|$, which has value $\sqrt{1-\left|\left\langle\psi_{1} \mid \psi_{2}\right\rangle\right|^{2}} \leq \sqrt{d}$.

Lemma 10 (Rigidity lemma). Let $d_{2} \geq d_{1} \geq 0$ be small constants. Suppose $|\psi\rangle,|\phi\rangle$, and $\sigma$ are defined as in Lemma 8, and $|\chi\rangle$ is defined as

$$
|\chi\rangle:=\frac{1}{\sqrt{R}} \sum_{j \in[R]}|j\rangle|\sigma(j)\rangle
$$

If $D(|\psi\rangle)=\frac{1}{\kappa}-d_{1}$ and $V(|\psi\rangle)=1-\frac{1}{\kappa}-d_{2}$, then $|\langle\chi \mid \psi\rangle|^{2} \geq 1-\kappa d_{1}-(\kappa+1) \sqrt{\kappa \cdot d_{2}}$.
Proof. By Lemma 8, we know that $|\langle\psi \mid \phi\rangle|^{2} \geq 1-\kappa \cdot d_{2}$. So by Fact 9, for any quantum state $|\mu\rangle$, $\left||\langle\mu \mid \phi\rangle|^{2}-|\langle\mu \mid \psi\rangle|^{2}\right| \leq \sqrt{\kappa \cdot d_{2}}$. We use this in two places. First, when $|\mu\rangle=|+\rangle$. Since $D(|\psi\rangle)=|\langle+\mid \psi\rangle|^{2}=$ $\frac{1}{\kappa}-d_{1}$, we have $|\langle+\mid \phi\rangle|^{2} \geq \frac{1}{\kappa}-d_{1}-\sqrt{\kappa \cdot d_{2}}$ by triangle inequality. Second, when $|\mu\rangle=|\chi\rangle$. Notice that

$$
|\langle\chi \mid \phi\rangle|^{2}=\kappa|\langle+\mid \phi\rangle|^{2} \geq 1-\kappa\left(d_{1}+\sqrt{\kappa \cdot d_{2}}\right) .
$$

Again by triangle inequality,

$$
|\langle\chi \mid \psi\rangle|^{2} \geq 1-\kappa\left(d_{1}+\sqrt{\kappa \cdot d_{2}}\right)-\sqrt{\kappa \cdot d_{2}} .
$$

Intuitively, Lemma 10 allows us to tune the probability of each test in the NP protocol. As we explain in the analysis (Section 3.3), if the probabilities of running Validity test and Density test are much higher than that for constraint tests, then if $d_{1}$ or $d_{2}$ is large, these two tests catch a "deceptive" quantum proof in soundness. This allows constraint tests to focus on the case of small $d_{1}$ and $d_{2}$; i.e. nearly rigid quantum proofs.

### 3.2 Constraint tests

We analyze the constraint tests on rigid quantum proofs, i.e. states of the form $|\psi\rangle=\frac{1}{\sqrt{R}} \sum_{j \in[R}|j\rangle\left|\vec{v}_{j}\right\rangle$. The verifier needs to check two properties:
(i) (satisfiability) For all $j \in[R]$, the assignment $\vec{v}_{j}$ satisfies $\mathcal{C}_{j}$.
(ii) (consistency) Each variable is assigned the same value when participating in different constraints.

One may ask why we even need to check for consistency. Couldn't we ask for the assignment of each variable $a:[N] \rightarrow \Sigma$, for example as the quantum proof $\frac{1}{\sqrt{N}} \sum_{i \in[N]}|i\rangle|a(i)\rangle$ ? The problem with this is checking satisfiability becomes difficult, since the assigned values are given in superposition. ${ }^{8}$

Instead, with a state $\frac{1}{\sqrt{R}} \sum_{j \in[R}|j\rangle\left|\vec{v}_{j}\right\rangle$, satisfiability is easy to verify: measure the first register (observing some $|j\rangle\left|\vec{v}_{j}\right\rangle$ ), and compute $\mathcal{C}_{j}\left(\vec{v}_{j}\right)$. Let $u_{s}$ be the number of unsatisfied constraints. The outcome is 1 with probability $1-\frac{u_{s}}{R}$.

But this form of quantum proof gives Merlin a new way to "deceive": for a given variable, send different values depending on the constraint! We prevent this by checking for consistency, similarly to the preprocessing step of [Din07] sometimes called regularization. As in [JW23, Section 7], we add "consistency constraints" to the CSP system $\mathcal{C}$ as follows: ${ }^{9}$

- For each variable $i \in[N]$, let $V_{i}$ represent the constraints that $i$ participates in.
- Fix a constant $d$. For each $i \in[N]$, draw a $d$-regular graph with vertices $V_{i}$ that is expanding. ${ }^{10}$
- Each edge $\left(j_{1}, j_{2}\right)$ of each expander $V_{i}$ represents a "consistency constraint", where we assert that the value of variable $i$ sent with constraint $j_{1}$ equals that sent with constraint $j_{2}$.

Using expander graphs allows us to prevent this kind of "deceptive" Merlin: either the proof fails many of the original constraints, or it fails many "consistency constraints". Let $u_{e}$ be the number of unsatisfied "consistency constraints" out of $R q \cdot \frac{d}{2}$ :

Claim 11 ([Din07, Lemma 4.1]). Consider $a(N, R, q, \Sigma)-C S P$ system $\mathcal{C}$, and apply regularization. If $\operatorname{val}(\mathcal{C})=1$, then all "consistency constraints" can be simultaneously satisfied. If $\operatorname{val}(\mathcal{C}) \leq \delta$, then the total number of unsatisfied constraints $\left(u_{s}+u_{e}\right)$ is at least $(1-\delta) R$.

How do we check these "consistency constraints"? Over the next few paragraphs, we construct a unitary related to permutations on the constraint graph. In completeness, the quantum proof is an eigenvector of this unitary, but in soundness, all rigid quantum proofs are detectably far (i.e. using a Hadamard test) from an eigenvector. We study the graph $\widetilde{G}$ with $R \cdot q$ vertices, where each vertex $(j, i)$ corresponds to a clause $j$ and a variable $i$ that participates in clause $j$. Let $\widetilde{G}$ be the union of all consistency edges created during regularization, i.e. $\left(j_{1}, j_{2}\right)$ for variable $i$ becomes the edge $\left(\left(j_{1}, i\right),\left(j_{2}, i\right)\right)$. Note that $\widetilde{G}$ contains a copy of each expander graph, so it is $d$-regular.

We now choose $d$ permutations. It is a classical fact that the adjacency matrix of any $d$-regular graph can always be decomposed to $d$ permutations. Let $\pi_{1}, \ldots, \pi_{d}$ be the decomposition of $\widetilde{G}$; recall that these are permutations on $V(\widetilde{G})$ where $|V(\widetilde{G})|=R \cdot q$. For each $k \in[d]$, we identify $\pi_{k}$ with a permutation on $[R] \times[N]$, where any $(j, i) \in[R] \times[N]$ that is not a vertex of $\widetilde{G}$ (i.e. variable $i$ does not participate in constraint $j$ ) is mapped to itself. ${ }^{11}$ Note that this map always preserves the variable $i \in[N]$; without loss of generality, we also identify $\pi_{k}$ with its restriction $[R] \times[N] \rightarrow[R]$. From here on, we use this last definition of $\pi_{k}$, which maps constraint $j_{1}$ that variable $i$ participates in to another constraint $j_{2}$ that variable $i$ participates in, and identity otherwise.

[^5]Now consider a rigid quantum proof, i.e. of the form $|\psi\rangle=\frac{1}{\sqrt{R}} \sum_{j \in[R]}|j\rangle\left|\vec{v}_{j}\right\rangle$. Since there are a polynomial number of variables and constraints, we can efficiently transform $|\psi\rangle$ to $\left|\phi^{\prime}\right\rangle$, where

$$
\left|\phi^{\prime}\right\rangle:=\frac{1}{\sqrt{q \cdot R}} \sum_{j \in[R]} \sum_{i \in \mathcal{C}_{j}}|j\rangle\left|\vec{v}_{j}\right\rangle|i\rangle\left|v_{j}(i)\right\rangle .
$$

Here, $i \in \mathcal{C}_{j}$ are the variables participating in $\mathcal{C}_{j}$, and $v_{j}(i)$ is the value of this variable according to $\vec{v}_{j}$.
We now would like to construct a unitary on $\left|\phi^{\prime}\right\rangle$ that maps $|j\rangle|i\rangle \mid$ value $\rangle$ to $\left|j^{\prime}\right\rangle|i\rangle \mid$ value $\rangle$ for some other constraint $j^{\prime}$ that $i$ participates in. In completeness, this unitary would leave the state unchanged. Notice that from the perspective of such a unitary, the second register containing $\left|\vec{v}_{j}\right\rangle$ is "junk". Fortunately, we can measure out the second register in the Hadamard basis, and reject if the outcome is not $|+\rangle$. All rigid states will observe outcome $|+\rangle$ with probability $\frac{1}{\kappa}$; one can see this by writing the second register in the Hadamard basis.

Suppose the observed outcome is $|+\rangle$; let us call the postselected state $|\phi\rangle$, where

$$
|\phi\rangle:=\frac{1}{\sqrt{q \cdot R}} \sum_{j \in R} \sum_{i \in \mathcal{C}_{j}}|j\rangle|i\rangle\left|v_{j}(i)\right\rangle .
$$

For each $k \in[d]$, we now implement the in-place transformation $\Pi_{k}$ according to $\pi_{k}:[R] \times[N] \rightarrow[R]$, where

$$
\Pi_{k}:|j\rangle|i\rangle\left|v_{j}(i)\right\rangle \rightarrow\left|\pi_{k}(j, i)\right\rangle|i\rangle\left|v_{j}(i)\right\rangle .
$$

Recall that the map $(j, i) \mapsto\left(\pi_{k}(j, i), i\right)$ is a permutation. Since we have access both to this permutation and its inverse, we can implement $\Pi_{k}$.

Note that in a satisfiable instance, $\Pi_{k}|\phi\rangle=|\phi\rangle$. By contrast, if $v_{j}(i) \neq v_{j^{\prime}}(i),\left|j^{\prime}\right\rangle|i\rangle\left|v_{j}(i)\right\rangle$ is orthogonal to $|\phi\rangle$. Hence, $\langle\phi| \Pi_{k}|\phi\rangle$ is the fraction of satisfied "consistency constraints" observed by $\pi_{k}$. We use the Hadamard test to measure this value, in a similar way to the Spectral test in [BFM22]. Note that unlike the swap test, the Hadamard test only uses one copy of a quantum state.

Definition 12 (Hadamard test). Let $|\psi\rangle$ be a quantum state and $U$ a unitary operator.

1. Prepend a control qubit to $|\psi\rangle$, to create $|0\rangle|\psi\rangle$.
2. Apply a Hadamard on the control qubit, to create $\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)|\psi\rangle$.
3. Apply $U$, controlled by the control qubit, to create $\frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle+|1\rangle U|\psi\rangle)$.
4. Apply a Hadamard on the control qubit, to create $\frac{1}{2}|0\rangle(|\psi\rangle+U|\psi\rangle)+\frac{1}{2}|1\rangle(|\psi\rangle-U|\psi\rangle)$.
5. Measure the control qubit, and accept if the output is 0 .

The success probability is then

$$
\frac{1}{4} \||\psi\rangle+U|\psi\rangle \|^{2}=\frac{1}{2}+\frac{1}{4}\langle\psi| U+U^{\dagger}|\psi\rangle=\frac{1}{2}+\frac{\operatorname{Re}\langle\psi| U|\psi\rangle}{2}
$$

We now can describe the constraint tests together:
(i) With probability $\frac{1}{q d \kappa+1}$, check satisfiability. This succeeds with probability $1-\frac{u_{s}}{R}$.
(ii) With probability $\frac{q d \kappa}{q d \kappa+1}$, generate $\left|\phi^{\prime}\right\rangle$, and measure the second register in the Hadamard basis. If the output state is not $|+\rangle$, reject. Otherwise, choose a random $k \in[d]$, and perform a Hadamard test with $\Pi_{k}$. This succeeds with probability $\frac{1}{\kappa}\left(\frac{1}{2}+\frac{1}{2} \mathbb{E}_{k}\left[\operatorname{Re}\langle\phi| \Pi_{k}|\phi\rangle\right]\right)=\frac{1}{\kappa}\left(1-\frac{u_{e}}{q d R}\right) .{ }^{12}$

[^6]The overall success probability of the constraint tests is

$$
\frac{1}{q d \kappa+1}\left(1-\frac{u_{s}}{R}\right)+\frac{q d \kappa}{q d \kappa+1}\left(\frac{1}{\kappa} \cdot\left(1-\frac{u_{e}}{q d R}\right)\right)=\frac{q d+1}{q d \kappa+1}-\frac{u_{e}+u_{s}}{R \cdot(q d \kappa+1)} .
$$

We now show a constant gap between completeness and soundness. In completeness, $u_{e}=u_{s}=0$, so $|\psi\rangle$ passes the constraint tests with probability $C^{\text {YES }}:=\frac{q d+1}{q d \kappa+1}$. In soundness, recall that $\operatorname{val}(\mathcal{C}) \leq \delta$, so by Claim 11, any rigid quantum proof passes the constraint tests with probability at most $C^{\text {YES }}-\frac{1-\delta}{q d \kappa+1}$. We now apply Lemma 10: any quantum proof that passes Density test and Validity test with probabilities too similar to that in completeness must pass the constraint tests with probability less than $C^{\text {YES }}$.
Corollary 13. In soundness, if $D(|\psi\rangle)=\frac{1}{\kappa}-d_{1}$ and $V(|\psi\rangle)=1-\frac{1}{\kappa}-d_{2}$, then

$$
C(|\psi\rangle) \leq C^{\mathrm{YES}}-\frac{1-\delta}{q d \kappa+1}+\left(\kappa d_{1}+(\kappa+1) \sqrt{\kappa \cdot d_{2}}\right)^{1 / 2}
$$

### 3.3 Analysis

In the protocol, Arthur applies Density test, Validity test, or constraint tests with probability $p_{1}, p_{2}, p_{3}$, respectively, where $p_{3}=1-p_{1}-p_{2}$.

We start by analyzing the success probability of the protocol in completeness. Here, $\operatorname{val}(\mathcal{C})=1$, and the quantum proof $|\psi\rangle=\frac{1}{R} \sum_{j}|j\rangle\left|\vec{v}_{j}\right\rangle$ is such that $\vec{v}_{j}$ is a satisfying assignment to the variables that participate in $\mathcal{C}_{j}$. The success probability for each test is exactly $\frac{1}{\kappa}, 1-\frac{1}{\kappa}$, and $C^{\mathrm{YES}}$, respectively. So the success probability of the protocol in completeness is $P_{\mathrm{YES}}=\frac{p_{1}}{\kappa}+p_{2}\left(1-\frac{1}{\kappa}\right)+p_{3} C^{\mathrm{YES}}$.

We now choose the probabilities $p_{1}, p_{2}, p_{3}$. Choose $\lambda:=\frac{1-\delta}{q d \kappa+1}$.

1. We first set a distance threshold $\varepsilon:=\frac{\lambda}{\Gamma}$ for a large enough constant $\Gamma(\kappa, q, d, \delta)$ satisfying

$$
(\kappa \varepsilon+(\kappa+1) \sqrt{\kappa \cdot \varepsilon})^{1 / 2} \leq \frac{\lambda}{2} .
$$

2. Let $Z:=\frac{1}{2}+1+\frac{\varepsilon}{4\left(1-C^{\mathrm{YES}}\right)}$. Then let

$$
p_{1}=\frac{1}{2} \cdot \frac{1}{Z} \quad p_{2}=\frac{1}{Z} \quad p_{3}=\frac{\varepsilon}{4\left(1-C^{\mathrm{YES}}\right)} \cdot \frac{1}{Z}
$$

Now we study soundness, i.e. when $\operatorname{val}(\mathcal{C}) \leq \delta$. We again denote the quantum proof as $|\psi\rangle$. We divide up the analysis into a few parts:

1. A quantum proof that is "too sparse" (i.e. $D(|\psi\rangle)=\frac{1}{\kappa}-d$ for any $d \geq \varepsilon$ ) is detected by Density test.

$$
\begin{aligned}
P_{\mathrm{NO}} & =p_{1}\left(\frac{1}{\kappa}-d\right)+p_{2} V(|\psi\rangle)+p_{3} C(|\psi\rangle) \\
& \leq P_{\mathrm{YES}}-p_{1} d+p_{3}\left(1-C^{\mathrm{YES}}\right) \\
& \leq P_{\mathrm{YES}}-p_{1} \varepsilon+p_{3}\left(1-C^{\mathrm{YES}}\right) \\
& =P_{\mathrm{YES}}-\frac{\varepsilon}{2 Z}+\frac{\varepsilon}{4 Z}=P_{\mathrm{YES}}-\frac{\varepsilon}{4 Z} .
\end{aligned}
$$

2. A quantum proof that is "too dense" (i.e. $D(|\psi\rangle)=\frac{1}{\kappa}+d$ for any $d \geq \varepsilon$ ) is detected by Validity test.

$$
\begin{aligned}
P_{\mathrm{NO}} & =p_{1}\left(\frac{1}{\kappa}+d\right)+p_{2} V(|\psi\rangle)+p_{3} C(|\psi\rangle) \\
& \leq p_{1}\left(\frac{1}{\kappa}+d\right)+p_{2}\left(1-\frac{1}{\kappa}-d\right)+p_{3} \\
& =P_{\mathrm{YES}}-\left(p_{2}-p_{1}\right) d+p_{3}\left(1-C^{\mathrm{YES}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \leq P_{\mathrm{YES}}-\left(p_{2}-p_{1}\right) \varepsilon+p_{3}\left(1-C^{\mathrm{YES}}\right) \\
& =P_{\mathrm{YES}}-\frac{\varepsilon}{2 Z}+\frac{\varepsilon}{4 Z}=P_{\mathrm{YES}}-\frac{\varepsilon}{4 Z}
\end{aligned}
$$

where the first inequality follows from Lemma 7.
3. A quantum proof that is "the right density" (i.e. $D(|\psi\rangle)=\frac{1}{\kappa}+d_{1}$ for $\left|d_{1}\right| \leq \varepsilon$ ) but far from "valid" $\left(V(|\psi\rangle)=1-\frac{1}{\kappa}-d_{2}\right.$ for $\left.d_{2} \geq \varepsilon\right)$ is detected by Validity test when $p_{2}>p_{1}$.

$$
\begin{aligned}
P_{\mathrm{NO}} & \leq p_{1}\left(\frac{1}{\kappa}+\left|d_{1}\right|\right)+p_{2}\left(1-\frac{1}{\kappa}-d_{2}\right)+p_{3} \\
& \leq P_{\mathrm{YES}}+p_{1}\left|d_{1}\right|-p_{2} d_{2}+p_{3}\left(1-C^{\mathrm{YES}}\right) \\
& \leq P_{\mathrm{YES}}-\left(p_{2}-p_{1}\right) \varepsilon+p_{3}\left(1-C^{\mathrm{YES}}\right) \\
& =P_{\mathrm{YES}}-\frac{\varepsilon}{4 Z}
\end{aligned}
$$

4. Lastly, a quantum proof that is nearly rigid (i.e. $D(|\psi\rangle)=\frac{1}{\kappa}+d_{1}$ and $V(|\psi\rangle)=1-\frac{1}{\kappa}-d_{2}$ for any $\left.\left|d_{1}\right|, d_{2} \leq \varepsilon\right)$ is detected by the constraint tests.

$$
\begin{aligned}
P_{\mathrm{NO}} & =p_{1}\left(\frac{1}{\kappa}+d_{1}\right)+p_{2}\left(1-\frac{1}{\kappa}-d_{2}\right)+p_{3} C(|\psi\rangle) \\
& \leq P_{\mathrm{YES}}+p_{1} d_{1}-p_{2} d_{2}+p_{3}\left(-\frac{1-\delta}{q d \kappa+1}+\left(\kappa d_{1}+(\kappa+1) \sqrt{\kappa \cdot d_{2}}\right)^{1 / 2}\right) \\
& \leq P_{\mathrm{YES}}+p_{1} d_{1}-p_{2} d_{2}-p_{3} \frac{\lambda}{2}
\end{aligned}
$$

The first inequality follows from Corollary 13 , and the second inequality holds by our choice of $\varepsilon$. Note that $d_{2} \geq 0$ by Lemma 6 . By Lemma 7, if $d_{1} \geq 0$, then $d_{1} \leq d_{2}$; otherwise $d_{1} \leq d_{2}$ trivially. So then

$$
P_{\mathrm{NO}} \leq P_{\mathrm{YES}}-\left(p_{2}-p_{1}\right) d_{2}-p_{3} \frac{\lambda}{2} \leq P_{\mathrm{YES}}-\frac{\varepsilon \lambda}{8\left(1-C^{\mathrm{YES}}\right) Z}
$$

Combining these cases proves the following result:
Theorem 14. Given an instance of $(1, \delta)$-GAPCSP, the $\mathrm{QMA}_{\log }^{+}$protocol succeeds with probability $P^{\mathrm{YES}}$ in completeness and at most $P^{\mathrm{YES}}-\Delta$ in soundness for some constants $1>P^{\mathrm{YES}}>\Delta>0$.

Corollary 15. There exist constants $1>P^{\mathrm{YES}}>\Delta>0$ such that $\mathrm{NP} \subseteq \mathrm{QMA}_{\log }^{+}$with completeness $P^{\mathrm{YES}}$ and soundness $P^{\mathrm{YES}}-\Delta$.

## 4 QMA $^{+}$protocol for NEXP

Our goal in this section is to modify the previous protocol to solve an NEXP-complete problem. Again by the PCP theorem, the succinct $(1, \delta)$-GapCSP problem with exponentially many variables and clauses is NEXP-complete. The succinctness allows us to efficiently describe the problem input. What remains is to ensure that the verifier's protocol is efficient. Previously, the unitary transformations were efficient because the verifier handled poly $(n)$-size graphs. Furthermore, the expanders used to check the equality constraints for each variable may have different sizes. Now that there can be exponentially many possibilities for the size of each cluster, naively applying the previous technique is not efficient. These challenges were addressed in [JW23] by considering a PCP construction for NEXP with strong properties.

Theorem 16 ([JW23]). There is a NEXP-hard $(1, \delta)$-GAPCSP instance for some $\left(N=2^{\text {poly }(n)}, R=\right.$ $\left.2^{\text {poly }(n)}, q=O(1), \Sigma=\{0,1\}\right)-C S P$ system $\mathcal{C}$ that is both $\tau$-strongly uniform for some constant $\tau$ and polylog(NR)-doubly explicit.

Informally, every constraint in a succinct CSP system must be computable in polynomial time. The doubly explicit property further requires the existence of efficient maps from variables to constraints and from constraints to variables. Intuitively, these maps allow us to efficiently implement the Hadamard test of the consistency checks.

We include the formal definition of these properties. Define $A d j_{\mathcal{C}}(j)$ to be the list of variables participating in $\mathcal{C}_{j}$, and $A d j_{V}(i)$ be the list of constraints that depend on variable $i$.

Definition 17 (Doubly explicit CSP). $A(N, R, q, \Sigma)$-CSP system $\mathcal{C}$ is $Z(N, R)$-doubly explicit if for all $i \in[N]$ and $j \in[R]$, the following are computable in time $Z(N, R)$ :
(i) Cardinality of $\operatorname{Adj}_{V}(i)$ and $\operatorname{Adj}_{\mathcal{C}}(j)$ for all $i \in[N]$ and $j \in[R]$.
(ii) $A d j_{\mathcal{C}}^{\text {ind }}:[R] \times[N] \rightarrow[q]$; if $i$ participates in $\mathcal{C}_{j}$, then $A d j_{\mathcal{C}}^{i n d}(j, i)=\imath$ is the index of $i$ in $A d j_{\mathcal{C}}(j)$.
(iii) $A d j_{\mathcal{C}}^{i d}:[R] \times[q] \rightarrow[N] ; \operatorname{Adj}_{\mathcal{C}}^{i d}(j, \imath)=i$ is the $\imath$-th variable of $\operatorname{Adj}_{\mathcal{C}}(j)$.
(iv) $A d j_{V}^{i n d}:[N] \times[R] \rightarrow[R]$; if i participates in $\mathcal{C}_{j}$, then $A d j_{V}^{i n d}(i, j)=\jmath$ is the index of $j$ in $A d j_{V}(i)$.
(v) $A d j_{V}^{i d}:[N] \times[R] \rightarrow[R] ; A d j_{V}^{i d}(i, \jmath)=j$ is the $\jmath$-th variable $A d j_{V}(i)$.

This property alone is not enough for efficient regularization: the verifier must know how to implement an expander of size $\left|A d j_{V}(i)\right|$ for all variables $i$. The strongly uniform property resolves this complication.

Definition 18 (Strongly uniform CSP). Let $\tau \in \mathbb{N}$. $A(N, R, q, \Sigma)$-CSP system $\mathcal{C}$ is $\tau$-strongly uniform if the variable set $[N]$ can be partitioned into at most $\tau$ different subsets $\bigcup_{y} V_{y}$ such that $\left|\operatorname{Adj} j_{V}(i)\right|=$ $\left|\operatorname{Adj}_{V}(j)\right|=n_{k}$ if $i$ and $j$ belong to the same part $V_{k}$. Furthermore, the part $k \in[\tau]$ can be determined in time polylog(NR).

A $\tau$-strongly uniform CSP system allows the verifier to use $\tau$ different (possibly exponential size) $d$-regular expanders. These can be constructed in polynomial time:

Theorem 19 (Doubly explicit expander graphs [Lub09, Alo21]). There is a constant d such that the following explicit constructions of expander graphs exist:

1. For every $n$, there is a d-regular graph on $n$ vertices.
2. For every prime $p>17$, there is a d-regular graph on $n=p\left(p^{2}-1\right)$ vertices, and the graph can be decomposed into $d$ permutations $\pi_{1}, \ldots, \pi_{d}$ that can each be evaluated in time polylog(n). ${ }^{13}$

Furthermore, the neighbors of each variable can be listed in $\operatorname{poly} \log (n)$, and the graphs have Cheeger constant at least 2.

With this theorem, the verifier can choose a large constant $n_{0}$, and use Construction 1 if $n_{i} \leq n_{0}$. Otherwise, the verifier can cover almost all $n_{i}$ vertices with an explicit expander using Construction 2.

Theorem 20 (Primes in short intervals [Che13]). There is an absolute constant $k_{0}$ such that for any integer $k>k_{0}$, there is a prime in the interval $\left[k-4 k^{2 / 3}, k\right]$.

We modify the protocol to expect the quantum proof $\left|p_{1}, p_{2}, \ldots, p_{\tau}\right\rangle \otimes \frac{1}{R} \sum_{j \in R}|j\rangle\left|\vec{v}_{j}\right\rangle$ such that each $p_{i} \in\left[\left\lfloor n_{i}^{1 / 3}\right\rfloor-4\left\lfloor n_{i}^{1 / 3}\right\rfloor^{2 / 3},\left\lfloor n_{i}^{1 / 3}\right\rfloor\right]$. The verifier measures the primes, and can always check that every $p_{i}$ is a prime number in the required range. The rest of the analysis is similar to the NP protocol, but regularized using these efficient expanders. We first explain how to efficiently implement the constraint tests, and then analyze the $\mathrm{QMA}^{+}$protocol.

[^7]
### 4.1 Efficient constraint tests

We show how to efficiently implement consistency checks that imply a version of Claim 11. Fix any vertex $i$. Let $n$ be the number of constraints that depend on $i$ and $p$ be the corresponding prime. Let $n_{0}$ be a large enough constant. If $n \leq n_{0}$, then use Construction $1 d$-regular expander to wire new copies of the vertex together, just as for NP. Otherwise, use Construction 2 to generate a $d$-regular expander graph of size $p\left(p^{2}-1\right) \in\left[n-O\left(n^{8 / 9}\right), n\right]$ that wires nearly all copies of the vertex together. Then add $d$ self-loops for the remaining vertices. The number of vertices with self loops is at most some $\eta n$ (for some small constant $\eta$ ) since $p\left(p^{2}-1\right) \geq n-O\left(n^{8 / 9}\right)$; we can make $\eta$ arbitrarily small by choosing a large enough $n_{0}$.

Claim 21 ([JW23]). Consider a $(N, R, q, \Sigma)-C S P$ system $\mathcal{C}$, and apply efficient regularization. If $\operatorname{val}(\mathcal{C})=1$, then all "consistency constraints" can be simultaneously satisfied. If $\operatorname{val}(\mathcal{C}) \leq \delta$, then the total number of unsatisfied constraints is at least $(1-\delta-q \eta) R$.

The analysis of Claim 21 is similar to Claim 11. The additional factor of $q \eta$ comes from the self-loop constraints; these can be satisfied without violating any "consistency constraint".

After measuring the primes, let the verifier act on the space $\mathbb{C}^{R} \otimes \mathbb{C}^{\kappa} \otimes \mathbb{C}^{N} \otimes \mathbb{C}^{|\Sigma|}$. We explicitly define the unitary operators that are used in the NP protocol. These definitions exactly match [JW23]. First, operator $\mathcal{A}$ expands the values from the list of values of variables involved in each constraint:

$$
\mathcal{A}:|j\rangle|v\rangle|0\rangle|0\rangle \rightarrow \frac{1}{q} \sum_{r=1}^{q}|j\rangle|\vec{v}\rangle\left|i_{r}\right\rangle\left|v_{r}\right\rangle,
$$

Here, $v$ is the list of values of variables involved in constraint $j, i_{r}$ is the $r$-th variable involved in $j$, and $v_{r}$ is the value of $i_{r}$ according to $v$. Next, define the permutation operators $\mathcal{M}_{k}$ for each $k \in[d]$ that implement the $d$ permutations of each efficiently constructed expander:

$$
\mathcal{M}_{k}:|j\rangle|v\rangle|i\rangle\left|v^{\prime}\right\rangle \rightarrow\left|\Pi_{k}(j, i)\right\rangle|v\rangle|i\rangle\left|v^{\prime}\right\rangle
$$

The last operation computes the constraints in superposition:

$$
\mathcal{B}|j\rangle|v\rangle|0\rangle \rightarrow|j\rangle|v\rangle\left|\mathcal{C}_{j}(v)\right\rangle
$$

Theorem 22 ([JW23]). $\mathcal{A}, \mathcal{B}, \mathcal{M}_{k}$ can be implemented by BQP circuits.

### 4.2 Analysis

The analysis is nearly the same as in the NP protocol, with the minor difference that the verifier also receives $p_{1}, \ldots, p_{\tau}$ in the quantum proof. The rigidity tests are unchanged. For the constraint tests, the verifier can use the explicit operators $\mathcal{A}, \mathcal{B}, \mathcal{M}_{k}$ :
(i) For satisfiability, the prover computes $\mathcal{B}|\psi\rangle|0\rangle$ and measures the second qubit in the standard basis.
(ii) For consistency, the prover computes $\mathcal{A}|\psi\rangle$, selects random $d \in[k]$, and uses $\mathcal{M}_{k}$ in Hadamard test.

We already know that for a quantum proof of the valid form, we can write the success probability as:

$$
C(|\psi\rangle)=\frac{q d+1}{q d \kappa+1}-\frac{u_{e}+u_{s}}{R \cdot(q d \kappa+1)}
$$

To be able to analyze soundness, all that is left is to reprove Corollary 13 to handle the subtle difference between Claim 21 and Claim 11. Let $D(|\psi\rangle)=\frac{1}{\kappa} \pm d_{1}$ and $V(|\psi\rangle)=1-\frac{1}{\kappa}-d_{2}$, then we can write:

$$
\begin{aligned}
C(|\psi\rangle) & =C^{\mathrm{YES}}-\frac{u_{e}+u_{s}}{R \cdot(q d \kappa+1)} \\
& \leq C^{\mathrm{YES}}-\frac{R \cdot(1-\delta-q \eta)}{R \cdot(q d \kappa+1)}+\left(\kappa d_{1}+(\kappa+1) \sqrt{\kappa \cdot d_{2}}\right)^{1 / 2}
\end{aligned}
$$

We can choose a small enough $\eta$ (via large enough $n_{0}$ ) so that $\eta q<\frac{1-\delta}{2}$.

Corollary 23. If $D(|\psi\rangle)=\frac{1}{\kappa} \pm d_{1}$ and $V(\psi)=1-\frac{1}{\kappa}-d_{2}$, then:

$$
C(|\psi\rangle) \leq C^{\mathrm{YES}}-\frac{1-\delta}{2(q d \kappa+1)}+\left(\kappa d_{1}+(\kappa+1) \sqrt{\kappa \cdot d_{2}}\right)^{1 / 2}
$$

For soundness, the same analysis of Section 3.3 goes through by reducing $\lambda$ by a factor of 2 ; this change comes from Corollary 23, which handles the extra $q \eta$ self-loop constraints. All together:
Theorem 24. Consider $(1, \delta)-G A P C S P$ with $a\left(N=2^{\text {poly }(n)}, R=2^{\text {poly }(n)}, q=O(1), \Sigma=\{0,1\}\right)$-CSP system that is polylog(NR)-doubly explicit and $O(1)$-strongly uniform. The $\mathrm{QMA}^{+}$protocol solves this problem with completeness $c$ and soundness $s$ for some constants $1>c>s>0$.

Corollary 25. There exist constants $1>c>s>0$ such that $\mathrm{NEXP} \subseteq \mathrm{QMA}_{c, s}^{+}$.

## 5 Subtle features of QMA $^{+}$

### 5.1 Promise symmetry matters

One can imagine restricting to proofs with non-negative amplitudes only in completeness. But this class is equal to QMA:
Fact 26. Consider the class $\mathrm{QMA}^{+^{\prime}}$, where the proof must have non-negative amplitudes only in completeness. Since subset states have non-negative amplitudes, $\mathrm{SQMA} \subseteq \mathrm{QMA}^{+} \subseteq \mathrm{QMA} . B y$ [GKS14], $S Q M A=\mathrm{QMA}^{+^{\prime}}=\mathrm{QMA}$.

Instead, QMA ${ }^{+}$also restricts the proof in soundness, which reduces the ways Merlin can "deceive" Arthur. This increases the power of the complexity class:

Corollary 27. Notice that $\mathrm{QMA}_{c, s}^{+{ }^{\prime}} \subseteq \mathrm{QMA}_{c, s}^{+}$for any choice of $0 \leq c, s \leq 1$, since any $\mathrm{QMA}_{c, s}^{+\prime}$ protocol is also a $\mathrm{QMA}_{c, s}^{+}$protocol. Since $\mathrm{QMA}=\mathrm{QMA}^{+}, \mathrm{QMA} \subseteq \mathrm{QMA}_{c, s}^{+}$whenever $c<1$ and $c-s \geq \frac{1}{p(n)}$ for any polynomial $p(n)$.

In general, suppose $\mathcal{R}$ is a set of quantum states that approximate all quantum states (i.e. an $\epsilon$-covering) by at least an inverse polynomial in number of qubits. Then QMA is equal to QMA restricted to $\mathcal{R}$ in completeness, and at most QMA restricted to $\mathcal{R}$ in both completeness and soundness.

Furthermore, classes that modify QMA only in completeness enjoy promise gap amplification through parallel repetition. This does not hold for promise-symmetric modifications. We provide a simple example of how parallel repetition fails to amplify the promise gap of $\mathrm{QMA}^{+}$:
Fact 28. Consider a Hermitian and positive semidefinite matrix $M$. Let $\|M\|_{+}:=\max _{|v\rangle \geq 0} \frac{\| M|v\rangle \|_{2}}{\||v\rangle \|_{2}}$ be the maximum value among real non-negative vectors. Then it is possible for $\|M \otimes M\|_{+}>\|M\|_{+}^{2}$.
Proof. Consider two qubits and the projector $M=\left|x_{-}\right\rangle\left\langle x_{-}\right|$, where $M$ projects into the Pauli-X basis (i.e. $\left.\left|x_{-}\right\rangle=\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)\right)$. Then $\|M\|_{+}=\frac{1}{\sqrt{2}}$, maximized at $|0\rangle$ or $|1\rangle$. But using the state $|\chi\rangle=\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$, $\|M \otimes M\|_{+} \geq \| M|\chi\rangle\left\|=\frac{1}{\sqrt{2}}>\right\| M \|_{+}^{2}$.

### 5.2 QMA $^{+}$at some constant gap equals QMA

Perhaps surprisingly, $\mathrm{QMA}_{c, s}^{+}$equals QMA for some constants $1>c>s>0$. This is because every quantum state can be approximated (up to a constant) by a quantum state without relative phase:
Proposition 29. $\mathrm{QMA}_{c, s}^{+} \subseteq \mathrm{QMA}_{c, 4 s}$.
Proof. Consider a problem in $\mathrm{QMA}_{c, s}^{+}$, and let $\Pi_{1}$ be its accepting operator. We will use the same circuit in QMA, and analyze the new completeness and soundness.

- (completeness) The same completeness proof is a valid proof for QMA, accepting with completeness $c$.
- (soundness) Recall that $\langle\chi| \Pi_{1}|\chi\rangle \leq s$ for any $|\chi\rangle$ with non-negative amplitudes.

Consider any state $|\psi\rangle$ with real (but possibly negative) amplitudes. Separate and normalize its positive entries and negative entries, i.e. $|\psi\rangle=\sqrt{p}\left|\psi_{+}\right\rangle-\sqrt{1-p}\left|\psi_{-}\right\rangle$. Notice that $\left\langle\psi_{+} \mid \psi_{-}\right\rangle=0$, so $|\psi\rangle$ is of unit norm. Then

$$
\begin{aligned}
\langle\psi| \Pi_{1}|\psi\rangle & =p\left\langle\psi_{+}\right| \Pi_{1}\left|\psi_{+}\right\rangle+(1-p)\left\langle\psi_{-}\right| \Pi_{1}\left|\psi_{-}\right\rangle-\sqrt{p(1-p)}\left(\left\langle\psi_{-}\right| \Pi_{1}\left|\psi_{+}\right\rangle+\left\langle\psi_{+}\right| \Pi_{1}\left|\psi_{-}\right\rangle\right) \\
& \leq p\left\langle\psi_{+}\right| \Pi_{1}\left|\psi_{+}\right\rangle+(1-p)\left\langle\psi_{-}\right| \Pi_{1}\left|\psi_{-}\right\rangle+2 \sqrt{p(1-p)} \sqrt{\left\langle\psi_{-}\right| \Pi_{1}\left|\psi_{-}\right\rangle\left\langle\psi_{+}\right| \Pi_{1}\left|\psi_{+}\right\rangle} \\
& \leq s+2 s \sqrt{p(1-p)} \leq 2 s,
\end{aligned}
$$

where the first inequality holds by Cauchy-Schwarz because $\Pi_{1}$ is positive semidefinite.
Similarly, consider any state with arbitrary amplitudes. Separate and normalize its real and imaginary entries, i.e. $|\phi\rangle=\sqrt{p^{\prime}}\left|\phi_{\mathbb{R}}\right\rangle+i \sqrt{1-p^{\prime}}\left|\phi_{i \mathbb{R}}\right\rangle$. Notice that $|\phi\rangle$ is still unit norm. By the same calculation, one finds that $\langle\phi| \Pi_{1}|\phi\rangle \leq 4 s$.

Corollary 30. For any $0<\varepsilon<0.2$, QMA $_{0.8+\varepsilon, 0.2}^{+}=$QMA.
Proof. $\mathrm{QMA}_{0.8+\varepsilon, 0.2}^{+} \subseteq$ QMA follows from Proposition 29. The other direction follows from Corollary 27.
This is a strange phenomenon: depending on the choice of constants $c>s, \mathrm{QMA}_{c, s}^{+}$could be as small as QMA and as large as NEXP! ${ }^{14}$ See Figure 2 for a pictorial description. An implication of our work is that assuming EXP $\neq$ NEXP, QMA ${ }^{+}$simply cannot be amplified.

## 6 Open questions

1. What is the relationship of QMA and QMA(2)? Our result does not immediately say anything about QMA and QMA(2). It only suggests that for QMA, the restriction of relative phase is maximally strong. For example, it is possible that QMA $(2)=$ NEXP; i.e. the restriction of entanglement across a fixed barrier may be just as powerful. In fact, showing $\mathrm{QMA}(2)=\mathrm{QMA}^{+}(2)$ is still an open route to proving $\mathrm{QMA}(2)=$ NEXP, but in light of this work, amplification for $\mathrm{QMA}^{+}(2)$ must crucially rely on the unentanglement promise.
2. Are other complexity classes sensitive to different constant-sized promise gaps? We show that for QMA,$+ s$, the parameter $\frac{c}{s}$ can be "tuned" to change the power of the class from QMA to NEXP (see also Figure 2). Do other complexity classes drastically change power with different promise gaps? One similar class is SBQP [Kup15], which equals $\mathrm{BQP}_{c, s}$ when $\frac{c}{s}=2$ (but unlike our work, $c$ and $s$ are exponentially small). However, when $\frac{c}{s}$ is allowed to be any number above $1, \mathrm{BQP}_{c, s}$ is equal to PP [DGF22]. Note that relative to oracles, SBQP is not closed under intersection, which was used to separate it from QMA [AKKT20].
3. State complexity vs. decision complexity? Although we prove that there are constants $c_{1}, s_{1}, c_{2}, s_{2}$ such that $\mathrm{QMA}_{c_{1}, s_{1}}^{+}=\mathrm{QMA}^{+}(2)_{c_{2}, s_{2}}$, we do not prove the existence of any product test (as in [HM13]). In fact, it is possible that no product test exists! This would show a separation between the complexity of decision problems and state synthesis problems; i.e. $\mathrm{QMA}^{+}=\mathrm{QMA}^{+}(2)$ but stateQMA ${ }^{+} \neq$ stateQMA ${ }^{+}(2)$. In fact, it is even possible that $\mathrm{QMA}=\mathrm{QMA}(2)$ but stateQMA $\neq$ stateQMA $(2)$. This inquiry can help us understand whether (or how) the power of unentanglement is useful when solving decision problems.
[^8]
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[^1]:    ${ }^{1}$ As noted before, restricting the state in completeness may not change the complexity class, but restricting the state in soundness can make the class more powerful, since the latter limits Merlin's adversarial strategies.
    ${ }^{2}$ This is because every state has constant overlap with some state without relative phase. See also Proposition 29.
    ${ }^{3}$ Formally, [JW23] studies states with non-negative amplitudes. Recall that the set of these states, up to global phase, are equivalent to states with no relative phase.
    ${ }^{4}$ Note that we use the intuition of rigidity in a more general context, where Arthur's tests, not a non-local game, enforce states of a certain form.

[^2]:    ${ }^{5}$ A subset state is a uniform superposition over a subset of all computational basis states. States with non-negative amplitudes are conical combinations of subset states.

[^3]:    ${ }^{6} \mathrm{QMA}^{+} \subseteq$ NEXP by directly simulating the quantum proof and verifier.

[^4]:    ${ }^{7}$ For simplicity, we denote the uniform superposition over all standard basis states by $|+\rangle$. The dimension is clear from the context.

[^5]:    ${ }^{8}$ There is a way around this limitation for CSP systems consisting of unique game constraints, where each (binary) constraint involving variables $i_{1}, i_{2}$ accepts exactly one $a\left(i_{2}\right)$ for each $a\left(i_{1}\right)$. See [JW23, Section 6] for more discussion.
    ${ }^{9}$ Note that since $N$ and $R$ are polynomially-sized, this process is efficient.
    ${ }^{10}$ For technical reasons of Claim 11, we require that the Cheeger constant is at least 2.
    ${ }^{11}$ These permutations (and their inverses) are all efficient because $N$ and $R$ are polynomially-sized.

[^6]:    ${ }^{12}$ Note that in our protocol, $\langle\phi| \Pi_{k}|\phi\rangle$ is always real because $|\phi\rangle$ and $\Pi_{k}$ have real values.

[^7]:    ${ }^{13}$ In fact, since these graphs are Cayley graphs, both the permutations and their inverses can be evaluated in time polylog $(n)$. We use both $\pi_{k}$ and $\pi_{k}^{-1}$ in the constraint tests to implement the unitary $\mathcal{M}_{k}$.

[^8]:    ${ }^{14}$ Note that the same phenomenon holds for $\mathrm{QMA}^{+}(2)$ and $\mathrm{QMA}(2)$ with nearly the same proof. This is why [JW23] was perceived as "just a constant gap away" from proving QMA(2) = NEXP.

